Compactness

Definition 1. A *cover* or a *covering* of a topological space X is a family $\mathcal{C}$ of subsets of X whose union is X. A *subcover* of a cover $\mathcal{C}$ is a subfamily of $\mathcal{C}$ which is a cover of X. An *open cover* of X is a cover consisting of open sets.

Definition 2. A topological space X is said to be *compact* if every open cover of X has a finite subcover. A subset S of X is said to be *compact* if S is compact with respect to the subspace topology.

Theorem 3. A subset S of a topological space X is compact if and only if every open cover of S by open sets in X has a finite subcover.

Proof. ($\Rightarrow$) Assume that S is a compact subset of a topology space X. Let $\{G_\alpha \mid \alpha \in \Lambda\}$ be a collection of open subset of X such that $S \subseteq \cup_{\alpha\in\Lambda} G_\alpha$. Let $O_\alpha = G_\alpha \cap S$ for each $\alpha \in \Lambda$. Then O_α is open in S for each $\alpha \in \Lambda$ and

$$\cup_{\alpha\in\Lambda} O_\alpha = \cup_{\alpha\in\Lambda}(G_\alpha \cap S) = (\cup_{\alpha\in\Lambda} G_\alpha) \cap S = S.$$

Thus $\{O_\alpha \mid \alpha \in \Lambda\}$ is an open cover of S. It follows that $\{O_\alpha \mid \alpha \in \Lambda\}$ contains a finite subcover $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$. Hence

$$S = \cup_{i=1}^n O_{\alpha_i} = \cup_{i=1}^n (O_{\alpha_i} \cap S) = (\cup_{i=1}^n G_{\alpha_i}) \cap S.$$

Thus $S \subseteq \cup_{i=1}^n G_{\alpha_i}$.

($\Leftarrow$) Let $\{O_\alpha \mid \alpha \in \Lambda\}$ be an open cover of (S, τ_s). Then, for each $\alpha \in \Lambda$, there exists an open set G_α in X such that $O_\alpha = G_\alpha \cap S$. Thus

$$S = \cup_{\alpha\in\Lambda} O_\alpha = \cup_{\alpha\in\Lambda}(G_\alpha \cap S) = (\cup_{\alpha\in\Lambda} G_\alpha) \cap S.$$

Thus $S \subseteq \cup_{\alpha\in\Lambda} G_\alpha$. Then there is a finite subset $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $S \subseteq \cup_{i=1}^n G_{\alpha_i}$. It follows that

$$S = (\cup_{i=1}^n G_{\alpha_i}) \cap S = \cup_{i=1}^n (G_{\alpha_i} \cap S) = \cup_{i=1}^n O_{\alpha_i}.$$

Hence (S, τ_s) is compact. □

Examples.

1. Any finite set is compact. In general, (X, τ), where τ is finite, is compact. In particular, an indiscrete space is compact.

2. Any infinite discrete space is not compact. In fact, if X is an infinite discrete space, then $\{\{x\} \mid x \in X\}$ is an open cover of X which has no finite subcover.

3. $\mathbb{R}$ is not compact. The class $\{(-n, n) \mid n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ which contains no finite subcover.

Theorem 4. A closed subset of a compact space is compact.

Proof. Let (X, τ) be a compact space and F a closed subset of X. Let $\mathcal{C} = \{G_\alpha \mid \alpha \in \Lambda\}$ be a family of open subsets of X such that $F \subseteq \cup_{\alpha\in\Lambda} G_\alpha$. Then

$$X = F \cup F^c = (\cup_{\alpha\in\Lambda} G_\alpha) \cup F^c.$$

Thus $\mathcal{C} \cup \{F^c\}$ is an open cover of X and hence it has a finite subcover $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}, F^c\}$, i.e. $X = (\cup_{i=1}^n G_{\alpha_i}) \cup F^c$. Since $F \subseteq X$, it follows that $F \subseteq \cup_{i=1}^n G_{\alpha_i}$. Thus $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ is a finite subfamily of $\mathcal{C}$ such that $F \subseteq \cup_{i=1}^n G_{\alpha_i}$. Hence, F is compact. $\square$

Theorem 5. If A is a compact subset of a Hausdorff space X and $x \notin A$, then x and A have disjoint neighborhoods.

Proof. By the Hausdorffness of X, for each $y \in A$, there are open neighborhoods U_y and V_y of x and y, respectively, such that $U_y \cap V_y = \varnothing$. Then $\{V_y \mid y \in A\}$ is an open cover for A, which is compact. Hence there are y_1, y_2, $\ldots$, $y_n \in A$ such that $A \subseteq \cup_{i=1}^n V_{y_i}$. Let $V = \cup_{i=1}^n V_{y_i}$ and $U = \cap_{i=1}^n U_{y_i}$. Then U and V are neighborhoods of x and A, respectively, and $U \cap V = \varnothing$. $\square$

Theorem 6. Any compact subset of a Hausdorff space is closed.

Proof. Let A be a compact subset of a Hausdorff space X. To show that A^c is open, let $x \in A^c$. By the previous theorem, there are neighborhoods U and V of x and A, respectively, such that $U \cap V = \varnothing$. It follows that $x \in U \subseteq V^c \subseteq A^c$. This shows that A^c is a neighborhood of x. Hence, A^c is a neighborhood of all its elements. It follows that A^c is open and that A is closed. $\square$

Theorem 7. A continuous image of a compact space is compact.

Proof. Let $f \colon X \to Y$ be a continuous function from a compact space X into a space Y. By Theorem 3, it is sufficient to assume that f maps X onto Y and we show that Y is compact. To see this, let $\{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover for Y. By continuity of f, it follows that $f^{-1}[G_\alpha]$ is open in X for each α and

$$\cup_{\alpha\in\Lambda} f^{-1}[G_\alpha] = f^{-1}[\cup_{\alpha\in\Lambda} G_\alpha] = f^{-1}[Y] = X.$$

Hence, $\{f^{-1}[G_\alpha] \mid \alpha \in \Lambda\}$ is an open cover for X. By the compactness of X, there are α_1, α_2, $\ldots$, $\alpha_n \in \Lambda$ such that $\cup_{i=1}^n f^{-1}[G_{\alpha_i}] = X$. Thus

$$Y = f[X] = f[\cup_{i=1}^n f^{-1}[G_{\alpha_i}]] = f[f^{-1}[\cup_{i=1}^n G_{\alpha_i}]] \subseteq \cup_{i=1}^n G_{\alpha_i}.$$

This shows that Y is compact. $\square$

Corollary 8. Let $f : X \to Y$ is a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. Let $f : X \to Y$ be a bijective continuous function. Assume that X is compact and Y is Hausdorff. To show that f is a homeomorphism, it suffices to show that f is a closed map. Let F be a closed subset of X. Then F is compact by Theorem 4. By Theorem 7, $f[F]$ is compact in the Hausdorff space Y. Hence, $f[F]$ is closed in Y by Theorem 6. □

Theorem 9. A continuous function of a compact metric space into a metric space is uniformly continuous.

Proof. Let $f\colon (X, d) \to (Y, \rho)$ be a continuous function between metric spaces. Assume that X is compact. To show that f is uniformly continuous, let $\varepsilon > 0$. By continuity of f, for each $x \in X$, there is a $\delta_x > 0$ such that

$$d(y, x) < \delta_x \implies \rho(f(y), f(x)) < \frac{\varepsilon}{2}.$$

Then $\{B_d(x, \frac{\delta_x}{2}) \mid x \in X\}$ is an open cover for a compact space X. Hence, we can choose $x_1, x_2, \ldots, x_n \in X$ such that $X = \cup_{i=1}^{n} B_d(x_i, \frac{\delta_{x_i}}{2})$. Choose $\delta = \frac{1}{2}\min\{d_{x_1}, d_{x_2}, \ldots, d_{x_n}\}$. Now, let $x, y \in X$ be such that $d(x, y) < \delta$. Then $x \in B_d(x_i, \frac{\delta_{x_i}}{2})$ for some i. Hence, $\rho(f(x), f(x_i)) < \frac{\varepsilon}{2}$. Moreover,

$$d(y, x_i) \leq d(y, x) + d(x, x_i) \leq \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}.$$

It follows that $\rho(f(y), f(x_i)) < \frac{\varepsilon}{2}$. By triangle inequality,

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \varepsilon.$$

This shows that f is uniformly continuous. □

Next we introduce a property that measures "boundedness" of a subset of a metric space. We say that a non-empty set A is bounded if $\operatorname{diam}(A) < \infty$, i.e., there is a constant $M > 0$ such that $d(x, y) \leq M$ for any $x, y \in A$. This definition of boundedness depends on the metric rather than the set itself. For example, $\mathbb{R}$ with the discrete metric is bounded even though in some sense $\mathbb{R}$ is "large". The following definition of total boundedness captures this spirit.

Definition 10. A metric space (X, d) is said to be *totally bounded* or *precompact* if for any $\varepsilon > 0$, there is a finite cover of X by sets of diameter less than ε.

Theorem 11. A metric space (X, d) is totally bounded if and only if for each $\varepsilon > 0$, X can be covered by finitely many ε-balls.

Proof. (⇒) Let $\varepsilon > 0$. Then there exist subsets A_1, A_1, ..., A_n of X such that $\operatorname{diam} A_i < \varepsilon$ for all $i \in \{1, 2, \dots, n\}$ and $\cup_{i=1}^n A_i = X$. We may assume that each A_i is non-empty and choose $x_i \in A_i$. If $x \in X$, then $x \in A_i$ for some i and hence, $d(x, x_i) \leq \operatorname{diam}(A_i) < \varepsilon$. This show that $X = \cup_{i=1}^n B_d(x_i, \varepsilon)$.

(⇐) Let $\varepsilon > 0$. Then there is a finite subset $\{x_1, x_2, \dots, x_n\}$ of X such that $X = \cup_{i=1}^n B_d(x_i, \frac{\varepsilon}{4})$. Let $A_i = B_d(x_i, \frac{\varepsilon}{4})$ for each $i \in \{1, 2, \dots, n\}$. For each a, $b \in A_i$, $d(a, b) \leq d(a, x_i) + d(x_i, b) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$. Hence, $\operatorname{diam}(A_i) < \varepsilon$. □

Theorem 12. A subspace of a totally bounded metric space is totally bounded.

Proof. Let (X, d) be a totally bounded metric space and Y a subspace of X. Let $\varepsilon > 0$. Then there exist A_1, A_2,..., $A_n \subseteq X$ such that $\operatorname{diam} A_i < \varepsilon$ for all $i \in \{1, 2, \dots, n\}$ and $\cup_{i=1}^n A_i = X$. Then $\operatorname{diam}(A_i \cap Y) \leq \operatorname{diam} A_i < \varepsilon$ for each $i \in \{1, 2, \dots, n\}$ and $\cup_{i=1}^n (A_i \cap Y) = (\cup_{i=1}^n A_i) \cap Y = X \cap Y = Y$. □

Theorem 13. Every totally bounded subset of a metric space is bounded.

Proof. Let (X, d) be a totally bounded metric space. Then there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of X such that $\cup_{i=1}^n B_d(x_i, 1) = X$. Let

$$K = \max\{d(x_i, x_j) \mid i, j \in \{1, 2, \dots, n\}\} + 2.$$

Let x, $y \in X$. Then $x \in B_d(x_i, 1)$ and $y \in B_d(x_j, 1)$ for some i, j. Thus

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 1 + d(x_i, x_j) + 1 \leq K,$$

i.e. $\operatorname{diam} X \leq K$, so X is bounded. □

In general, the converse is not true. The space $\mathbb{R}$ with the discrete metric is bounded because $d(x, y) \leq 1$ for all $x, y \in \mathbb{R}$, but it cannot be covered by a finitely many balls of radius $\frac{1}{2}$. However, it is true for $\mathbb{R}^n$ with the usual metric.

Theorem 14. A bounded subset of $\mathbb{R}^n$ is totally bounded.

Proof. We will prove this theorem for the case $n = 1$. The proof for the case $n > 1$ is similar but slightly more complicated.

By Theorem 12, it suffices to prove that a closed interval $[a, b]$ is totally bounded. Let $\varepsilon > 0$. Choose an $n \in N$ such that $(b - a)/n < \varepsilon$. For $i = 0, 1, \dots, n$, let

$$x_i = a + i\frac{b - a}{n}.$$

Then $[a, b] = \cup_{i=1}^n [x_{i-1}, x_i]$ and $\operatorname{diam}([x_{i-1}, x_i]) = x_i - x_{i-1} < \varepsilon$. □

Theorem 15. A metric space is totally bounded if and only if every sequence in it has a Cauchy subsequence.

Proof. ($\Leftarrow$) Assume that (X, d) is not tally bounded. Then there is an $\varepsilon > 0$ such that X cannot be covered by finitely many balls of radius ε. Let $x_1 \in X$. Then $B_d(x_1, \varepsilon) \neq X$, so we can choose $x_2 \in X - B_d(x_1, \varepsilon)$. In general, for each $n \in \mathbb{N}$, we can choose $x_{n+1} \in X - \cup_{i=1}^{n} B_d(x_i, \varepsilon)$. If $m > n$, then $x_m \notin B_d(x_n, \varepsilon)$, and thus $d(x_m, x_n) \geq \varepsilon$. It is easy to see that (x_n) has no Cauchy subsequence.

($\Rightarrow$) Assume that (X, d) is totally bounded and let (x_n) be a sequence in (X, d). Set $B_0 = X$. There exist $A_{11}, A_{12}, \dots, A_{1n_1} \subseteq X$ such that

$$\operatorname{diam} A_{1i} < 1 \text{ for all } i \in \{1, 2, \dots, n_1\} \text{ and } \bigcup_{i=1}^{n_1} A_{1i} = X.$$

At least one of these A_{1i}'s, called B_1, must contain infinitely many terms of (x_n). Let $(x_{11}, x_{12}, \dots)$ be a subsequence of (x_n) which lies entirely in B_1. Since $B_1 \subseteq X$, it is totally bounded. There are $A_{21}, \dots, A_{2n_2} \subseteq B_1$ such that

$$\operatorname{diam} A_{2i} < \frac{1}{2} \text{ for all } i \in \{1, 2, \dots, n_2\} \text{ and } \bigcup_{i=1}^{n_2} A_{2i} = B_1.$$

At least one of these A_{2i}'s, called B_2, must contain infinitely many terms of (x_{1n}). Then $\operatorname{diam} B_2 < \frac{1}{2}$ and we can choose a subsequence $(x_{21}, x_{22}, x_{23}, \dots)$ of $(x_{11}, x_{12}, x_{13}, \dots)$ in B_2. We can continue this process. To make this argument more precise, we will give an inductive construction.

Assume that we can choose $B_i \subseteq B_{i-1}$ with $\operatorname{diam} B_i < \frac{1}{i}$ and a subsequence $(x_{i1}, x_{i2}, x_{i3}, \dots)$ of $(x_{(i-1)1}, x_{(i-1)2}, x_{(i-1)3}, \dots)$ in B_i for all $i \in \{1, 2, \dots, k-1\}$. Since $B_{k-1} \subseteq B_{k-2} \subseteq \dots B_1 \subseteq B_0 = X$, B_{k-1} is totally bounded. Then there exist $A_{k1}, A_{k2}, \dots, A_{kn_k} \subseteq B_{k-1}$ such that

$$\operatorname{diam} A_{kj} < \frac{1}{k} \text{ for all } j \in \{1, 2, \dots, n_k\} \text{ and } \bigcup_{i=1}^{n_k} A_{ki} = B_{k-1}.$$

At least one of these A_{ki}'s, called B_k, must contain infinitely many terms of $(x_{k-1,n})$. Hence $B_k \subseteq B_{k-1}$, $\operatorname{diam} B_k < \frac{1}{k}$ and we can choose a subsequence $(x_{k1}, x_{k2}, x_{k3}, \dots)$ of $(x_{(k-1)1}, x_{(k-1)2}, x_{(k-1)3}, \dots)$ which lies entirely in B_k.

Now we choose the diagonal elements $(x_{11}, x_{22}, x_{33}, \dots)$ from the above subsequences. This is to guarantee that the index of the chosen subsequence is strictly increasing. To see that it is a Cauchy subsequence of (x_n), let $\varepsilon > 0$. Choose an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Let m, $n \in \mathbb{N}$ be such that m, $n \geq N$. Then $x_{mm} \in B_m \subseteq B_N$ and $x_{nn} \in B_n \subseteq B_N$. Thus

$$d(x_{mm}, x_{nn}) \leq \operatorname{diam} B_N < \frac{1}{N} < \varepsilon.$$

Hence (x_{nn}) is a Cauchy subsequence of (x_n). □

Definition 16. A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

Theorem 17. A metric space X is sequentially compact if and only if it is complete and totally bounded.

Proof. ($\Rightarrow$) Let (X, d) be a sequentially compact space. By Theorem 15, it is totally bounded because a convergent sequence is a Cauchy sequence. To see that it is complete, let (x_n) be a Cauchy sequence in X. Since X is sequentially compact, it has a convergent subsequence. Hence, (x_n) is convergent. It follows that X is complete.

($\Leftarrow$) Assume that (X, d) is totally bounded and complete. To see that X is sequentially compact, let (x_n) be a sequence in X. By Theorem 15, it has a Cauchy subsequence (x_{n_k}). Since (X, d) is complete, (x_{n_k}) is convergent. Hence, (x_n) has a convergent subsequence. This shows that X is sequentially compact. $\square$

Definition 18. Let $\mathcal{C}$ be a cover of a metric space X. A *Lebesgue number* for $\mathcal{C}$ is a positive number λ such that any subset of X of diameter less than or equal to λ is contained in some member of $\mathcal{C}$.

Remark. If λ is a Lebesgue number, then so is any $\lambda' > 0$ such that $\lambda' \leq \lambda$.

Example. Let $X = (0, 1) \subseteq \mathbb{R}$ and $\mathcal{C} = \{(\frac{1}{n}, 1) \mid n \geq 2\}$. Then $\mathcal{C}$ is an open cover for X, but it has no Lebesgue number. To see this, let $\lambda > 0$. Choose a positive integer n such that $\frac{1}{n} < \lambda$. Let $A = (0, \frac{1}{n}) \subseteq X$. Then $\operatorname{diam}(A) = \frac{1}{n} < \lambda$, but A is not contained in any member of $\mathcal{C}$. Hence $\mathcal{C}$ has no Lebesgue number.

Theorem 19. Every open cover of a sequentially compact metric space has a Lebesgue number.

Proof. Let $\mathcal{C}$ be an open cover of a sequentially compact metric space (X, d). Suppose that $\mathcal{C}$ does not have a Lebesgue number. Then for each $n \in \mathbb{N}$ there exists a subset B_n of X such that

$$\operatorname{diam}(B_n) \leq \frac{1}{n} \quad \text{and} \quad B_n \not\subseteq G \quad \text{for all } G \in \mathcal{C}.$$

For each $n \in \mathbb{N}$, choose $x_n \in B_n$. Since X is sequentially compact, the sequence (x_n) contains a convergent subsequence (x_{n_k}). Let $x \in X$ be the limit of this subsequence. Then $x \in G_0$ for some $G_0 \in \mathcal{C}$. Since G_0 is open, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq G_0$. Since (x_{n_k}) converges to x, there exists an integer N such that

$$d(x_{n_k}, x) < \frac{\varepsilon}{2} \quad \text{for any } k \geq N.$$

Choose $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\varepsilon}{2}$. Let $K = \max\{M, N\}$. Then $n_K \geq K \geq M$. Hence,

$$d(x_{n_K}, x) < \frac{\varepsilon}{2} \quad \text{and} \quad x_{n_K} \in B_{n_K}.$$

Moreover,

$$\operatorname{diam}(B_{n_K}) \leq \frac{1}{n_K} < \frac{\varepsilon}{2}.$$

For any $y \in B_{n_K}$, we have

$$d(x, y) \leq d(x, x_{n_K}) + d(x_{n_K}, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $y \in B_d(x, \varepsilon)$. Then $B_{n_K} \subseteq B_d(x, \varepsilon) \subseteq G_0$. This contradicts the fact that $B_{n_K} \nsubseteq G$ for all $G \in \mathcal{C}$. □

Definition 20. A space X is said to satisfy the *Bolzano-Weierstrass property* if every infinite subset has an accumulation point in X.

Theorem 21. In a metric space X, the following statements are equivalent:

(a) X is compact;

(b) X has the Bolzano-Weierstrass property;

(c) X is sequentially compact;

(d) X is complete and totally bounded.

Proof. We have already proved $(c) \Leftrightarrow (d)$ in Theorem 17.

$(a) \Rightarrow (b)$. Let (X, d) be a compact metric space and S an infinite subset of X. Suppose that S has no accumulation point. Then for each $x \in X$ there exists an open neighborhood V_x such that $V_x \cap (S - \{x\}) = \varnothing$. Hence, $\mathcal{C} = \{V_x \mid x \in X\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$ of $\mathcal{C}$. Since each ball contains at most one point of S, $X = \cup_{i=1}^{n} V_{x_i}$ contains finitely many points of S. Hence, S is finite, contrary to the hypothesis.

$(b) \Rightarrow (c)$. Assume that X has the Bolzano-Weierstrass property. To show that X is sequentially compact, let (x_n) be a sequence in X.

Case I. The set $S = \{x_n \mid n \in \mathbb{N}\}$ is finite. Then there is $a \in X$ such that $x_n = a$ for infinitely many n's. Choose $n_1 = \min\{n \in \mathbb{N} \mid x_n = a\}$ and for any $k \geq 2$, let

$$n_k = \min(\{n \in \mathbb{N} \mid x_n = a\} - \{n_1, n_2, \ldots, n_{k-1}\}).$$

Then (x_{n_k}) is a constant subsequence of (x_n) and hence is convergent.

Case II. The set $S = \{x_n \mid n \in \mathbb{N}\}$ is infinite. By the assumption, S has an accumulation point x (in X). For each $n \in \mathbb{N}$, $B_d(x, \frac{1}{n}) \cap (S - \{x\}) \neq \varnothing$; in fact, $B_d(x, \frac{1}{n}) \cap (S - \{x\})$ is an infinite set. Let

$$n_1 = \min\{n \in \mathbb{N} \mid x_n \in B_d(x, 1) \cap (S - \{x\})\} \qquad \text{and}$$
$$n_k = \min(\{n \in \mathbb{N} \mid x_n \in B_d(x, \tfrac{1}{k}) \cap (S - \{x\})\} - \{n_1, \dots, n_{k-1}\}),$$

for any $k \geq 2$. Then (x_{n_k}) is a subsequence of (x_n) such that $d(x_{n_k}, x) < \frac{1}{k}$ for each $k \in \mathbb{N}$. Hence (x_{n_k}) converges to x.

$(c) \Rightarrow (a)$. Assume that (X, d) is a sequentially compact metric space. Let $\mathcal{C}$ be an open cover for X. Hence $\mathcal{C}$ has a Lebesgue number $\lambda > 0$. Moreover, X is totally bounded. Thus there exist $A_1, A_2, \dots, A_n \subseteq X$ such that $X = \cup_{i=1}^n A_i$ and $\operatorname{diam}(A_i) \leq \lambda$ for each i. Hence for each $i \in \{1, 2, \dots, n\}$, there exists $G_i \in \mathcal{C}$ such that $A_i \subseteq G_i$. Thus $X = \cup_{i=1}^n G_i$. This shows that $\mathcal{C}$ has a finite subcover. Hence X is compact. □

Theorem 22 (Heine-Borel). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. By Theorem 21, a metric space is compact if and only if it is complete and totally bounded. Hence, if a subset of $\mathbb{R}^n$ is compact, then it is closed and bounded. (In fact, we can prove directly that compactness implies a set being closed and bounded without using Theorem 21.) Conversely, let A be a closed and bounded subset of $\mathbb{R}^n$. By Theorem 14, A is totally bounded. Since A is closed subset of $\mathbb{R}^n$, which is complete, A is also complete. Hence, A is compact. □

Corollary 23 (Extreme Value Theorem). A real-valued continuous function on a compact space has a maximum and a minimum.

Proof. Assume that $f\colon X \to \mathbb{R}$ is a continuous function and X is compact. By Theorem 7, $f[X]$ is a compact subset of $\mathbb{R}$. Hence, $f[X]$ is closed and bounded. Let $a = \inf f[X]$ and $b = \sup f[X]$. Since $f[X]$ is closed, a and b are in $f[X]$. Thus a and b are maximum and minimum of $f[X]$, respectively. □

Theorem 24 (Bolzano-Weierstrass). Every bounded infinite subset of $\mathbb{R}^n$ has at least one accumulation point (in $\mathbb{R}^n$).

Proof. Let A be a bounded infinite subset of $\mathbb{R}^n$. Since A is bounded, it is contained in some closed cube $I_n = [-n, n] \times \cdots \times [-n, n]$. Since I_n is closed and bounded, it is compact by Heine-Borel theorem. Since A is an infinite subset of a compact set I_n, it must have an accumulation point (in $\mathbb{R}^n$) by Theorem 21. □

Theorem 25. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}^n$. Hence it is contained in some closed cube $I_n = [-n, n] \times \cdots \times [-n, n]$. Since I_n is closed and bounded, it is compact by Heine-Borel theorem. Then I_n is sequentially compact. This implies that (x_n) has a convergent subsequence. $\square$