Completion of a Metric Space

Definition. A *completion* of a metric space (X, d) is a pair consisting of a complete metric space (X^*, d^*) and an isometry $\varphi\colon X \to X^*$ such that $\varphi[X]$ is dense in X^*.

Theorem 1. *Every metric space has a completion.*

Proof. Let (X, d) be a metric space. Denote by $\mathcal{C}[X]$ the collection of all Cauchy sequences in X. Define a relation $\sim$ on $\mathcal{C}[X]$ by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

It is easy to see that this is an equivalence relation on $\mathcal{C}[X]$. Let X^* be the set of all equivalence classes for $\sim$:

$$X^* \;=\; \{\, [(x_n)] \;:\; (x_n) \in \mathcal{C}[X] \,\}.$$

Define $d^*\colon X^* \times X^* \to [0, \infty)$ by

$$d^*\left([(x_n)], [(y_n)]\right) = \lim_{n\to\infty} d(x_n, y_n),$$

where $[(x_n)]$, $[(y_n)] \in X^*$. To show that d^* is well-defined, let (x'_n) and (y'_n) be two Cauchy sequences in X such that $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$. Then

$$\lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d(y_n, y'_n) = 0.$$

By the triangle inequality,

$$\begin{aligned} d(x_n, y_n) &\le d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \quad \text{and} \\ d(x'_n, y'_n) &\le d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n). \end{aligned}$$

Hence,

$$|d(x_n, y_n) - d(x'_n, y'_n)| \;\le\; d(x_n, x'_n) + d(y_n, y'_n) \;\longrightarrow\; 0.$$

Since both $(d(x_n, y_n))$ and $(d(x'_n, y'_n))$ are convergent, this shows that

$$\lim_{n\to\infty} d(x_n, y_n) \;=\; \lim_{n\to\infty} d(x'_n, y'_n).$$

Thus d^* is well-defined.

Next, we show that d^* is a metric on X^*. Let $[(x_n)]$, $[(y_n)]$, $[(z_n)] \in X^*$. Then

$$d^*([(x_n)], [(y_n)]) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0 \Leftrightarrow (x_n) \sim (y_n) \Leftrightarrow [(x_n)] = [(y_n)].$$

Also,

$$d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = d^*([(y_n)], [(x_n)]).$$

Since $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$,

$$\lim_{n\to\infty} d(x_n, z_n) \ \leq \ \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n).$$

Thus

$$d^*([(x_n)], [(z_n)]) \ \leq \ d^*([(x_n)], [(y_n)]) + d^*([(y_n)], [(z_n)]).$$

Hence d^* is a metric on X^*.

For each $x \in X$, let $\hat{x} = [(x, x, \dots)] \in X^*$, the equivalence classes of the constant sequence $(x, x, \dots)$. Define $\varphi\colon X \to X^*$ by $\varphi(x) = \hat{x}$. Then for any x, $y \in X$,

$$d^*(\varphi(x), \varphi(y)) = d^*(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Hence φ is an isometry from X into X^*. To show that $\varphi[X]$ is dense in X^*, let $x^* = [(x_n)] \in X^*$ and let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for any m, $n \geq N$, $d(x_m, x_n) < \dfrac{\varepsilon}{2}$. Let $z = x_N$. Then $\hat{z} \in \varphi[X]$ and

$$d^*(x^*, \hat{z}) \ = \ \lim_{n\to\infty} d(x_n, z) \ = \ \lim_{n\to\infty} d(x_n, x_N) \ \leq \ \frac{\varepsilon}{2} \ < \ \varepsilon.$$

Thus $\hat{z} \in B_{d^*}(x^*, \varepsilon) \cap \varphi[X]$. Hence, $\varphi[X]$ is dense in X^*.

Finally we show that (X^*, d^*) is complete. To establish this, we apply the following lemma of which proof is left as an exercise:

Lemma. *Let (X, d) be a metric space and A a dense subset such that every Cauchy sequence in A converges in X. Prove that X is complete.*

Hence, it suffices to show that every Cauchy sequence in the dense subspace $\varphi[X]$ converges in X^*. Let $(\widehat{z}_k)$ be a Cauchy sequence in $\varphi[X]$, where each $\widehat{z}_k$ is represented by the Cauchy sequence $(z_k, z_k, \dots)$. Since φ is an isometry,

$$d(z_n, z_m) \ = \ d^*(\widehat{z}_n, \widehat{z}_m) \qquad \text{for each } m,\ n.$$

Hence, $(z_1, z_2, z_3, \dots)$ is a Cauchy sequence in X. Let $z^* = [(z_1, z_2, z_3, \dots)] \in X^*$. To show that $(\widehat{z}_k)$ converges to z^*, let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $d(z_k, z_n) < \dfrac{\varepsilon}{2}$ for any k, $n \geq N$. Hence, for each $k \geq N$,

$$d^*(\widehat{z}_k, z^*) \ = \ \lim_{n\to\infty} d(z_k, z_n) \ \leq \ \frac{\varepsilon}{2} \ < \ \varepsilon.$$

This shows that $(\widehat{z}_k)$ converges to a point z^* in X^* and that X^* is complete. □

Theorem 2. *A completion of a metric space is unique up to isometry. More precisely, if* $\{\varphi_1, (X_1^*, d_1^*)\}$ *and* $\{\varphi_2, (X_2^*, d_2^*)\}$ *are two completions of* (X, d)*, then there is a unique isometry* f *from* X_1^* *onto* X_2^* *such that* $f \circ \varphi_1 = \varphi_2$*.*

Proof. Since φ_1 is an isometry, φ_1 is 1-1. Thus $\varphi_1^{-1} \colon \varphi_1[X] \to X$ is an isometry from $\varphi_1^{-1}[X]$ onto X. Since φ_2 is an isometry from X onto $\varphi_2[X] \subseteq X_2^*$, it follows that $\varphi_2 \circ \varphi_1^{-1} \colon \varphi_1[X] \to \varphi_2[X]$ is a surjective isometry. Let $h = \varphi_2 \circ \varphi_1^{-1}$. Then

$$h \circ \varphi_1 = (\varphi_2 \circ \varphi_1^{-1}) \circ \varphi_1 = \varphi_2 \circ (\varphi_1^{-1} \circ \varphi_1) = \varphi_2 \circ i_X = \varphi_2.$$

Hence there exists a unique isometry f from X_1^* into X_2^* which is an extension of h. For each $x \in X$,

$$f \circ \varphi_1(x) = f(\varphi_1(x)) = h(\varphi_1(x)) = h \circ \varphi_1(x) = \varphi_2(x).$$

Thus $f \circ \varphi_1 = \varphi_2$. Similarly, there exists a unique isometry g from X_2^* into X_1^* such that $g \circ \varphi_2 = \varphi_1$. Therefore

$$g \circ f \circ \varphi_1 = g \circ \varphi_2 = \varphi_1 \qquad \text{and} \qquad f \circ g \circ \varphi_2 = f \circ \varphi_1 = \varphi_2.$$

Hence $g \circ f = i_{\varphi_1[X]}$ and $f \circ g = i_{\varphi_2[X]}$. Since $\varphi_1[X]$ is dense in X_1^*, we have $g \circ f = i_{X_1^*}$. Similarly, $f \circ g = i_{X_2^*}$. Thus $f = g^{-1}$. Hence, f is a unique isometry from X_1^* onto X_2^* such that $f \circ \varphi_1 = \varphi_2$. □