14. Time Series Analysis: Serial Correlation

Read Wooldridge (2013), Chapter 12

Outline

I. Properties of OLS with Serially Correlated Errors

II. Testing for Serial Correlation

III. Correcting for Serial Correlation

IV. ARCH (Autoregressive conditional heteroskedasticity) model

V. Heteroskedasticity in Time Series

I. Properties of OLS with Serially Correlated Errors

• If there is the problem of serial correlation, i.e.,
  \[ E(u_t u_s | x_t, x_s) \neq 0 \quad \text{for } s \neq t \]
  \[ E(u_t u_{t-1}) \neq 0 \quad \text{for } AR(1) \]

• Serial Correlation means that errors are correlated. (See Graphs)

• Static and finite distributed lag models often have serially correlated errors.
Unbiasedness and Serial Correlation

- **Unbiasedness (TS.1 - TS.3)**
  In the presence of serial correlation, OLS estimators are unbiased in finite samples.

- **Consistency (TS.1'-TS.3')**
  In the presence of serial correlation, OLS estimators are consistent in large samples.

- **If there is the problem of serial correlation**, the Gauss Markov Theorem no longer holds.
  - Note that Gauss Markov Theorem (TS.1'-TS.5') requires no heteroskedasticity (TS.4') and no serial correlation (TS.5').

Errors follow the AR(1) process

Assume TS.1-TS.4 hold

3) **Variance of** $\hat{\beta}_1$ **is**

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{SST_x} + \frac{2\sigma^2}{SST_x} \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \rho^i x_i x_{t+j}$$

4) **s.e.$(\hat{\beta}_1)$** **is invalid.**
   - Usually, OLS s.e. will underestimate the true s.e. when estimating using Eviews.
   - Thus, the usual t statistic is invalid. Moreover, F and LM statistics are also invalid.

II. Testing for Serial Correlation

**Methods**

1. **When regressors are strictly exogenous.**
   - t-test for AR(1)
   - Durbin-Watson Statistic

2. **When regressors are not strictly exogenous.**
   - t-test for AR(1)
   - F-test for AR(q)
   - Breusch-Godfrey LM serial correlation Test

**Strict exogeneity implies that $u_t$ is uncorrelated with regressors for all time periods.**
1. Test for AR(1) serial correlation with strictly exogenous regressors

- Consider the model
  \[ y_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \ldots + \beta_k x_{tk} + u_t \]
  \[ u_t = \rho u_{t-1} + e_t \]  \[ u_t \] are weakly dependent

where (i) \( \{ u_t \} \) follow the AR(1) process and (ii) \( \{ e_t \} \) ~ iid(0, \( \sigma_e^2 \)) and \(|\rho| < 1\)

- \( H_0: \rho = 0 \) or the null hypothesis is that there is no serial correlation. If the null is true, then TS.5 is true.

Use residuals for errors

- Given AR(1) process
  \[ u_t = \rho u_{t-1} + e_t \]

- Estimation:
  \[ \hat{u}_t = \rho \hat{u}_{t-1} + e_t \]

  to find the \( \hat{\rho} \) (coefficient of \( \hat{u}_{t-1} \)) and its corresponding t statistic.

- Rejection rule: If \( t > 2 \), then \( H_0: \rho = 0 \) is rejected. We conclude that there is first-order serial correlation.

Notes on t-test for AR(1) serial correlation

1) This is a large sample test.
   \( \hat{\rho} \) is the consistent estimator of \( \rho \).
   (estimation error in \( \hat{\rho} \))

2) Practical and statistical significance.
   If \( t > 2 \) but \( \hat{\rho} \) is small, then OLS inference procedures are not far off.

3) This t test detects serial correlation of adjacent terms, AR(1) in \( u_t \)

Example: Static Phillips Model

- Phillips curve shows the tradeoff between inflation and unemployment.

  \[ inf_t = \beta_0 + \beta_1 unem_t + u_t \]

- Step 1: Estimation output
  \[ \hat{inf}_t = 1.423 + .468 unem \]
  \[ (s.e. \{1.72\} \{.289\}) \]
  \[ [t] \{.828\} \{1.617\} \]
  n=49 \( R^2=.053 \) \( R^2_{bar}=.032 \)

- Interpret the coefficient on \( unem \).
Static Phillips

Step 2: Use $\tilde{u}$ (residuals) to test for serial correlation.

Eviews: Proc/Make Residual Series (resid01)

Step 3: Test for serial correlation.

Regress $\tilde{u}$ on $\tilde{u}_{t-1}$ to find $\beta$ and $t$ value.

$$\text{resid01} = 0.573 \text{resid01(-1)}$$

($\text{s.e.}$) $(0.115013)$

($t$) $(4.9797)$

• Is there any problem of serial correlation?

Step 1: Regress $\text{inf}$ on $\text{unem}$

Step 2: Eviews: Proc/Make Residual Series (named resid01)

Dependent Variable: INF
Method: Least Squares
Sample: 1 49
Included observations: 49

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.42361</td>
<td>1.719015</td>
<td>0.828154</td>
<td>0.4118</td>
</tr>
<tr>
<td>UNEM</td>
<td>0.467626</td>
<td>0.289126</td>
<td>1.617376</td>
<td>0.1125</td>
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R-squared 0.052723
Adjusted R-squared 0.032568
S.D. dependent var 4.108163
S.E. of regression 5.160264
Durbin-Watson stat 0.8027

Step 3: Obtain $\tilde{u}$ (resid01) to test for serial correlation by regressing resid01 on resid01(-1)

Dependent Variable: RESID01
Method: Least Squares
Sample(adjusted): 2 49
Included observations: 48 after adjusting endpoints

<table>
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<tr>
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<td>RESID01(-1)</td>
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<td>0.115013</td>
<td>4.979738</td>
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</table>

R-squared 0.344633
Adjusted R-squared 0.344633
S.D. dependent var 3.046154
S.E. of regression 5.237479
Durbin-Watson stat 1.351045

Example: Expectations Augmented Phillips Curve

• $\text{inf}_t - \text{inf}_t^e = \beta_1 (\text{unem}_t - \mu_0) + e_t$

$\mu_0$: natural rate of unemployment
$\text{inf}_t^e$: expected rate of inflation
$\text{unem}_t - \mu_0$: cyclical unemployment
$\text{inf}_t - \text{inf}_t^e$: unanticipated inflation
$e_t$: supply shock (error term)

• Adaptive Expectation theory says that the expected value of current inflation depends on recently observed inflation.

Let $\text{inf}_t^e = \text{inf}_{t-1}$

$\Delta \text{inf}_t = \beta_0 + \beta_1 \text{unem}_t + e_t$
Results: Expectations Augmented Phillips Curve

**Step 1:** regress $\Delta \inf_t$ on $\unem_t$

$\Delta \inf_t = 3.03 - 0.543\unem_t$

<table>
<thead>
<tr>
<th>(s.e)</th>
<th>(1.38)</th>
<th>(0.280)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[t-stat]</td>
<td>[2.2]</td>
<td>[-2.35]</td>
</tr>
<tr>
<td>n = 48</td>
<td>$R^2 = 0.108$</td>
<td>$R^2$ bar = 0.088</td>
</tr>
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</table>

*Interpret the coefficient of $\unem_t$.

**Step 2:** Use $\hat{\unem}$ (residuals) to test for serial correlation – resid02

**Step 3:** Test for serial correlation.

Regress $\hat{\unem}_{t+1}$ on $\hat{\unem}_t$ to find $\rho$ and t-stat

$\text{resid02} = -0.0357 \text{resid02}(-1)$

<table>
<thead>
<tr>
<th>(s.e)</th>
<th>{.123}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[t-stat]</td>
<td>[-.289]</td>
</tr>
</tbody>
</table>

• Is there any problem of serial correlation?

---

### Eviews: Proc/Make Residual Series (name resid02)

**Step 1:** Estimation output: Regress $\Delta \inf_t$ on $\unem_t$

<table>
<thead>
<tr>
<th>Dependent Variable: INF-INF(-1)</th>
<th>Method: Least Squares</th>
</tr>
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<tbody>
<tr>
<td>Sample(adjusted): 2 49</td>
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<td>3.030581</td>
<td>1.37681</td>
<td>2.201161</td>
<td>0.0328</td>
</tr>
<tr>
<td>UNEM</td>
<td>-0.54259</td>
<td>0.230156</td>
<td>-2.357475</td>
<td>0.0227</td>
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R-squared 0.107796 Mean dependent var -0.10625
Adjusted R-squared 0.0884 S.D. dependent var 2.566926
S.E. of regression 2.450843 Akaike info criterion 4.671515
Sum squared resid 276.3051 Schwarz criterion 4.749682
Log likelihood -110.1164 F-statistic 5.557689
Durbin-Watson stat 1.769484 Prob(F-statistic) 0.02271

**Step 2:** Eviews: Proc/Make Residual Series (name resid02)

**Step 1:** Use $D(\inf)$ in place of $\Delta \inf_t$ in Eviews

<table>
<thead>
<tr>
<th>Dependent Variable: D(INF)</th>
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Durbin-Watson stat 1.769484 Prob(F-statistic) 0.02271
**Step 2:** Test for Serial Correlation.
Regress $\hat{u}_t$ on $\hat{u}_{t-1}$

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<td>RESID02(-1)</td>
<td>-0.03566</td>
<td>0.123104</td>
<td>-0.289695</td>
<td>0.7734</td>
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</tbody>
</table>

R-squared: -0.00744  
Adjusted R-squared: -0.00744  
Mean dependent var: 0.194241  
S.D. dependent var: 2.038688  
S.E. of regression: 4.290946  
Sum squared resid: 192.6093  
Schwarz criterion: 4.330311  
Log likelihood: -99.8372  
Durbin-Watson stat: 1.827911

**Durbin-Watson Test**

- The Durbin-Watson Test is valid under classical assumptions only. (TS.1-6 except TS.5)
- Durbin Watson (DW) Statistic is
  \[ DW = \frac{\sum_{i=2}^{n} (\hat{u}_i - \hat{u}_{i-1})^2}{\sum_{i=1}^{n} \hat{u}_i^2} \]
  \[ \rho = \frac{\sum_{i=2}^{n} \hat{u}_i \hat{u}_{i-1}}{\sum_{i=1}^{n} \hat{u}_i^2} \]
- Simple Algebra shows that DW and $\rho$ are closely linked
  \[ DW \approx 2(1- \rho) \] (*)
  using the fact that $\sum \hat{u}_i^2 = \sum \hat{u}_{i-1}^2$ and with moderate sample sizes, equation (*) is pretty close.

**Examples**

- **Example:** Static Phillips Model
  \[ \ln f = 1.423 + .468 \text{unem} \]
  n=49  \[ R^2 = .053 \quad R^2\text{bar} = .032 \]
  Durbin-Watson stat = 0.80  
  Table (n=50, k=1, $\alpha=1\%$): $d_L = 1.324$; $d_U = 1.403$  
  DW < $d_L$ or .80 < 1.324; there is an evidence of serial correlation

- **Example:** Expectations Augmented Phillips Model
  \[ \Delta \ln f = 3.03 - 0.543 \text{umem} \]
  n = 48  \[ R^2 = 0.108 \quad R^2\text{bar} = 0.088 \]
  Durbin-Watson stat = 1.80  
  Table (n=45, k=1, $\alpha=5\%$): $d_L = 1.48$; $d_U = 1.57$  
  DW > $d_U$ or 1.80 > 1.57; there is no evidence of serial correlation.
Advantages and Disadvantages

- **Advantage of DW Test:** exact sampling distribution for DW can be tabulated.
- **Disadvantages of DW Test:** DW is valid under the full set of CLM assumptions.

Advantages of t Test:
- t statistic is asymptotically valid even without TS.6 assumption.
- t statistic is valid in the presence of heteroskedasticity that depends on the $x_{ij}$.

2. Testing for AR(1) Serial Correlation without Assuming Strict Exogeneity

- Durbin (1970). Given a model,
  
  $y_t = \beta_0 + \beta_1 x_{t1} + \ldots + \beta_k x_{tk} + u_t$

  where $x_t$ could contain lagged dependent variables (e.g., $y_{t-1}$).

- Durbin suggested to run the regression of $\hat{u}_t$ on $x_{t1}, \ldots, x_{tk}, \hat{u}_{t-1}$ and obtain $\hat{\rho}$, the coefficient of $\hat{u}_{t-1}$ and its corresponding t value.

- Null Hypothesis: $H_0: \rho = 0$
  
  t-test: Rejection Rule: if $t > 2$, then we reject $H_0$.

Example: Static Phillips Revisited

- $infr = \beta_0 + \beta_1 unem_t + u_t$

  - **Step 1:** Estimation output
    
    $\hat{infr} = 1.423 + .468 unem$  
    (s.e) (1.72) (.289)  
    $[t] \quad [.828] \quad [1.617]$  
    n=49 $R^2=.053 \quad R^2 bar=.032$

  - **Step 2:** Use $\hat{u}$ (resid01) to test for serial correlation.
Example: Static Phillips Revisited

- **Step 3**: Regress $\hat{u}_t$ on $\text{unem}$ and $\hat{u}_{t-1}$

  \[ \text{resid01} = 2.157 - .3929\text{unem} + .6449\text{resid01(-1)} \]

  \[ t = (5.247) \]

  \[ n=48, R^2=.380747 \]

- Interpretation:
  1) Do we reject the null hypothesis according to $t$ test?
  2) **LM Test**: $LM = (n-q)*R^2-squared = (49-1)*.380747 = 18.29$
      $c=6.63$ is the 99th percentile in the distribution of $\chi^2_{1}$.

---

**Step 1**: Regress $\text{inf}$ on $\text{unem}$

**Step 2**: Eviews: Proc/Make Residual Series (named resid01)

- **Step 3**: Regress $\hat{u}_t$ on $\text{unem}$ and $\hat{u}_{t-1}$ to find $\hat{\rho}$ in the case that $\text{unem}$ is *not* strictly exogenous.

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<td>-0.392975</td>
<td>0.247479</td>
<td>-1.587912</td>
<td>0.1193</td>
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<th>Prob.</th>
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<tr>
<td>$\text{C}$</td>
<td>2.157068</td>
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<td>0.15</td>
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<tr>
<td>$\text{UNEM}$</td>
<td>-0.392975</td>
<td>0.247479</td>
<td>-1.587912</td>
<td>0.1193</td>
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<td>$\text{RESID01(-1)}$</td>
<td>0.644904</td>
<td>0.122912</td>
<td>5.246876</td>
<td>0</td>
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<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
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<tr>
<td>$\text{C}$</td>
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<td>1.847909</td>
<td>0.0711</td>
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<tr>
<td>$\text{UNEM}$</td>
<td>-0.47356</td>
<td>0.24753</td>
<td>-1.913156</td>
<td>0.062</td>
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<td>$\text{RESID(-1)}$</td>
<td>0.659484</td>
<td>0.125004</td>
<td>5.275690</td>
<td>0</td>
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</table>

- Breusch-Godfrey Serial Correlation LM Test:
  - F-statistic: 27.83291
  - Probability: 0.000003
  - Obs*R-squared: 18.47161
  - Probability: 0.000017

- Test Equation:
  - Dependent Variable: RESID

  Presample missing value lagged residuals set to zero.

---

**Eviews**: the case of nonstrictly exogeneous regressors in the “equation” window (step 1), choose View/Residual Tests/Serial Correlation LM Test.

In the “lag Specification” box, type in “1” for one lag in residuals.
Testing for Higher Order Serial Correlation

- Test serial correlation in the autoregressive of order q or AR(q)
  \[ u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \ldots + \rho_q u_{t-q} + e_t \]
- Test the null hypothesis
  \[ H_0: \rho_1 = \rho_2 = \ldots = \rho_q = 0 \]
- We can run the regression of
  \[ \tilde{u}_t \text{ on } x_{it}, \ldots, x_{it}, u_{t-1}, \ldots, u_{t-q} \]
- Compute the F test for joint significance of \( u_{t-1}, \ldots, u_{t-q} \)
or the LM statistic is
  \[ LM = (n-q)R^2_u \sim \chi^2_q \]
This is called the \textit{Breusch-Godfrey test for AR(q)} serial correlation

Example: Expectations Augmented Phillips Curve Revisited

- Regress \( D(\text{inf}) \) on \( unem \)
  \[ \Delta \tilde{\text{inf}}_t = 3.03 - 0.543 \text{unem}_t, \quad n = 48 \quad R^2 = 0.108 \quad R^2 \text{ bar} = 0.088 \]
- Suppose we wish to test
  \[ H_0: \rho_1 = \rho_2 = 0 \] in the AR(2) model
  \[ u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \delta \text{unem}_t + e_t \]
- Run the regression of
  \[ \text{resid}_c \text{ unem resid}_(-1) \text{ resid}_(-2) \]

Eviews: Expectations Augmented Phillips Curve Revisited

- In the “equation” window (step 1), choose View/Residual Tests/ Serial Correlation LM Test.
  In the “lag Specification” box, type in “2” for one lag in residuals.
- \[ H_0: \rho_1 = \rho_2 = 0 \]  
  F-statistic 4.409 (p-value = .017979)  
  LM-statistic 8.0136 (p-value = .018191)  
We reject the null hypothesis, so there is strong evidence of AR(2) serial correlation. Compare to AR(1)!
III. Correcting for serial correlation

Remedial measures: After detecting serial correlation, we now learn how to fix the problem.

1) Assume strictly exogenous regressors
   1.1 known \( \rho \): find generalized least squares (GLS) estimators
   1.2 use estimated \( \rho \) - feasible GLS (FGLS)
      • Cochrane-Orcutt Method
      • Iterated Cochrane-Orcutt Method
      • Differencing

2) Assume without strictly exogeneity
   – find SC-robust standard error (heteroskedasticity and autocorrelation consistent standard error)

1.1) Find best linear unbiased estimators (BLUE) in the AR(1) model

• assume the errors \( \{u_t\} \) follow the AR(1) model:
  \[ u_t = \rho u_{t-1} + e_t \]
  where \( e_t \sim i.i.d(0, \sigma_e^2) \) and \( |\rho| < 1 \)

• consider the simple regression model
  \[ y_t = \beta_0 + \beta_1 x_t + u_t \]
  note that \( \text{Var}(u_t) = \sigma_e^2 / (1 - \rho^2) \)

Find First Observation in GLS Estimation

• we can show that for \( t \geq 2 \),
  \[ \tilde{y}_t = \beta_0 \tilde{x}_{t0} + \beta_1 \tilde{x}_{t1} + e_t \]
  where quasi-differenced data \( (|\rho| < 1) \) are
  \[ \tilde{y}_t = y_t - \rho y_{t-1}; \tilde{x}_{t0} = 1 - \rho; \tilde{x}_{t1} = x_t - \rho x_{t-1} \]

• equation for \( t=1 \) is
  \[ y_1 = \beta_0 + \beta_1 x_1 + u_1 \] (*)

Using the fact that \( \text{Var}(u_t) = \sigma_e^2 / (1 - \rho^2) \), we multiply (*) by \( (1 - \rho^2)^{1/2} \) to get errors with the same variance \( \sigma_e^2 \), i.e.,

\[ (1 - \rho^2)^{1/2} y_t = (1 - \rho^2)^{1/2} \beta_0 + \beta_1 (1 - \rho^2)^{1/2} x_t + (1 - \rho^2)^{1/2} u_t \]

Generalized difference equation

• for the multiple regression model, we can easily find the generalized difference equation for \( j=2, ..., k \):
  \[ \tilde{y}_t = \beta_0 \tilde{x}_{t0} + \beta_1 \tilde{x}_{t1} + \ldots + \beta_k \tilde{x}_{tk} + \text{error} \]
  where \( \tilde{y}_t = y_t - \rho y_{t-1}; \tilde{x}_{tj} = x_{tj} - \rho x_{t-1} \)
GLS estimators are BLUE

- Since the errors \( \{e_t\} \) are assumed to be an i.i.d. sequence with mean zero and constant variance.
  
  \[
  \begin{align*}
  E(e_t) &= 0 \\
  \text{var}(e_t) &= \sigma^2 \\
  E(e_t e_{t-1}) &= 0
  \end{align*}
  \]

- GLS estimators are BLUE since the errors in the transformed equation are serially uncorrelated and homoskedastic. Thus, t and F statistics are valid.

1.2. FGLS Estimation with AR(1) errors

- The “\( \rho \)” is a population parameter. We rarely know the true value of \( \rho \) in practice.

- We know how to get the a consistent estimator of \( \rho \). That is, run the regression of \( \hat{u}_t \) on \( \hat{u}_{t-1} \) to obtain the sample counterpart \( \hat{\rho} \).

FGLS Estimation with AR(1) errors

- We now can run the model with \( \hat{\rho} \):

  \[
  \tilde{y}_t = \beta_0 \tilde{x}_{t0} + \beta_1 \tilde{x}_{t1} + \ldots + \beta_k \tilde{x}_{tk} + \text{error}
  \]

  \[
  \tilde{y}_t = y_t - \hat{\rho} y_{t-1}
  \]

  \[
  \tilde{x}_{t0} = 1 - \hat{\rho}
  \]

  \[
  \tilde{x}_{tj} = x_{tj} - \hat{\rho} x_{t-1,j} \quad j=2,\ldots,k
  \]

  This results in the feasible GLS (FGLS) estimator of the \( \beta_j \).

- FGLS estimators are not unbiased, but consistent.

- FGLS estimators are asymptotically more efficient than the OLS estimator when \( \{u_t\} \) is the AR(1) process.

Cochrane-Orcutt Estimation

- Comparing OLS and FGLS

  \[
  y_t = \beta_0 + \beta_1 x_t + u_t
  \]

<table>
<thead>
<tr>
<th>OLS-Assumptions</th>
<th>FGLS-Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cov((x_t,u_t))=0</td>
<td>Cov((x_t,u_t))=0</td>
</tr>
<tr>
<td>Cov((x_t,x_{t-1}),(u_t))=0</td>
<td>-</td>
</tr>
</tbody>
</table>

\[
\]
CO and PW Estimation

- If we omit the first observation, FGLS estimation is called the **Cochrance-Orcutt (CO)** estimation.
- If we include the first observation, FGLS estimation is called the **Prais-Winsten (PW)** estimation.

**Example: Static Phillips Model**

- **Step 1:** OLS output
  \[ \text{inf} = 1.423 + .468 \text{unem} \]
  Since \( \bar{\rho} = .573 \) and \( t = 4.98 \) (t>2), there is very strong evidence of serial correlation.

- **Step 2:** Cochrance-Orcutt Estimation: \( \bar{\rho} = .573 \)
  \[
  \text{inf}_t - \bar{\rho} \text{inf}_{t-1} = (1 - \bar{\rho}) \beta_0 + \beta_1 (\text{unem}_t - \text{unem}_{t-1}) + e_t
  \]
  \[
  \text{inf}_t - .573 \text{inf}_{t-1} = 5.51 - 0.28 (\text{unem}_t - .573 \text{unem}_{t-1})
  \]
  \[
  n=48 \quad R^2 = .016
  \]

**Cochrane-Orcutt: \( \bar{\rho} = .573 \)**

**OLS and Cochrane-Orcutt Compared**

1. There is inflation-unemployment tradeoff.
2. s.e.'s under the CO method are uniformly higher.
3. The CO method reports one fewer observation than OLS.
4. R-squared in the OLS is valid whereas R-squared in CO estimation is not meaningful.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>OLS</th>
<th>Cochrane-Orcutt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>1.42</td>
<td>5.51</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(1.72)</td>
<td>(2.04)</td>
</tr>
<tr>
<td>{t}</td>
<td>{83}</td>
<td>{271}</td>
</tr>
<tr>
<td>Unem</td>
<td>0.47</td>
<td>-0.28</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.289)</td>
<td>(0.32)</td>
</tr>
<tr>
<td>{t}</td>
<td>{1.62}</td>
<td>{-0.869}</td>
</tr>
<tr>
<td>(Observations)</td>
<td>49</td>
<td>48</td>
</tr>
<tr>
<td>R-Squared</td>
<td>0.0527</td>
<td>0.016</td>
</tr>
</tbody>
</table>

Note that 5.51 = (1 - \( \bar{\rho} \)) \( \bar{\rho} \) = (1 - .573) \( \bar{\rho} \).
Thus, \( \bar{\rho} = 12.9 \)
Iterative Cochrane-Orcutt Method

- In practice, the Cochrane-Orcutt method is used in an iterative scheme.
- Steps for iterative Cochrane-Orcutt Method
  
  **Step 1:** use $\hat{\rho}$ to find run the following regression.
  
  \[ y_t - \hat{\rho} y_{t-1} = \beta_0 (1 - \hat{\rho}) + \beta_1 (x_t - \hat{\rho} x_{t-1}) + e_t \]

  **Step 2:** We can obtain a new set of residuals and a new estimate of $\rho$.
  
  **Step 3:** Repeat step (1) again until the estimate of $\rho$ changes by very little.

- By iterative method above, we can find that $\hat{\rho} = .774$ for the static Phillips model.

- Eviews: Regress the following: $y \times x_t$ and AR(1). The coefficient of AR(1) is the iterative Cochrane-Orcutt estimate of $\rho$.

Iterated Cochrane-Orcutt: $\hat{\rho} = .774$

Inf. -.774*Inf(-1) (1-.774) Unem. -.774*Unem(-1)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>7.583459</td>
<td>2.531239</td>
<td>2.995947</td>
<td>0.0044</td>
</tr>
<tr>
<td>UNEM</td>
<td>-0.66534</td>
<td>0.361515</td>
<td>-1.840412</td>
<td>0.0723</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.774051</td>
<td>0.104367</td>
<td>7.416633</td>
<td>0</td>
</tr>
</tbody>
</table>

Dependent Variable: INF
Sample(adjusted): 2-49
Included observations: 48 after adjusting endpoints
Estimation settings: tol=0.00010, derive=accurate mixed (linear)
Convergence achieved after 6 iterations

- By iterative method above, we can find that $\hat{\rho} = .774$ for the static Phillips model.

1. Negative Relationship between inflation and unemployment.
2. R-squared
3. Observations
4. Note that
   - $\hat{\rho}_0 = -7.58$
   - $\hat{\rho}_0 = -7.58$
   - $\hat{\rho}_0 = -33.5$

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>OLS</th>
<th>Cochrane-Orcutt</th>
<th>Iterated Cochrane-Orcutt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>1.42</td>
<td>5.51</td>
<td>-7.58</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(.72)</td>
<td>(2.04)</td>
<td>(2.38)</td>
</tr>
<tr>
<td>(t)</td>
<td>1.92</td>
<td>2.71</td>
<td>3.19</td>
</tr>
<tr>
<td>Unem</td>
<td>0.47</td>
<td>-0.28</td>
<td>-0.665</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(.289)</td>
<td>(.32)</td>
<td>(.319)</td>
</tr>
<tr>
<td>(t)</td>
<td>1.62</td>
<td>-0.869</td>
<td>-2.08</td>
</tr>
<tr>
<td>(Observations)</td>
<td>48</td>
<td>48</td>
<td>48</td>
</tr>
<tr>
<td>R-Squared</td>
<td>0.052</td>
<td>0.016</td>
<td>0.086</td>
</tr>
</tbody>
</table>
2. Differencing and Serial Correlation

- One benefit of differencing: It is a transformation for making an integrated process weakly dependent $[I(0)]$.
- Consider the model
  \[ y_t = \beta_0 + \beta_1 x_t + u_t \]
- We have learned as follows.
  1) If $y_t$ or $x_t$ are integrated of order one $I(1)$, OLS test statistics can be misleading.
  2) Differencing can mitigate or may get rid of serially correlated errors.

Differencing in Phillips model

- Do differencing the static Phillips model and estimate.
  \[ \Delta y_t = \beta_1 \Delta x_t + \epsilon_t \]
  \[ \Delta y_t = y_t - y_{t-1} \]
- Results:
  1. There is tradeoff between inflation and unemployment.
  2. $\hat{\rho} = -0.03047$ (p-value= 0.8444); there is no evidence of serial correlation after differencing.
  3. Note that $\Delta y_t$ and $\Delta x_t$ may contain a unit root (how do we know?), so differencing might be needed to produce $I(0)$ variables.

Results from differencing:

1. There is a tradeoff between inflation and unemployment.
Breusch-Godfrey Serial Correlation LM Test:

<table>
<thead>
<tr>
<th>F-statistic</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.038992</td>
<td>0.844353</td>
</tr>
</tbody>
</table>

Obs*R-squared: 0.041556 Probability 0.838469

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Presample missing value lagged residuals set to zero.

Variable Coefficient Std. Error t-Statistic Prob.
C -0.00063 0.352175 -0.001793 0.9986
D(UNEM) 0.016852 0.328852 0.051245 0.9594
RESID(-1) -0.03047 0.154296 -0.197465 0.8444

R-squared 0.000866 Mean dependent var 1.13E-17
Durbin-Watson stat 1.809113 Prob(F-statistic) 0.980701

2. \( \hat{\beta} = -0.03047 \) (p-value= 0.8444); there is no evidence of serial correlation after differencing.

Eviews: View/Residual Tests/Serial Correlation LM Test of AR(1)

3. Serial Correlation-Robust Inference After OLS

- Reasons for taking this approach:
  #1. Explanatory variables may not be strictly exogeneous; thus, FGLS is not even consistent.
  #2. In most application of FGLS, the errors are assumed to follow AR(1). It may be better to compute standard errors for the OLS estimates that are robust to more general forms of serial correlation.

- Given a model
  \( y_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \ldots + \beta_k x_{tk} + u_t \)

Find \( se(\hat{\beta}_1) = \left(\frac{se(\hat{\beta}_1)}{\hat{\sigma}}\right)^2 \sqrt{\hat{v}} \)

- Write auxiliary regression for \( x_{t1} \) as
  \( x_{t1} = \delta_0 + \delta_2 x_{t2} + \ldots + \delta_k x_{tk} + r_{it} \)
  \( r_{it} = x_{t1} - \delta_0 - \delta_2 x_{t2} - \ldots - \delta_k x_{tk} \)

- \( \hat{\delta} = \hat{\beta} \hat{\sigma}_t \)

- \( g = 4(n/100)^{1/2} \) (Newley West and Eviews)
  - Annual: \( g=1,2 \)
  - Quarter: \( g=4,8 \)
  - Monthly: \( g=12,24 \)
SERIAL CORRELATION-ROBUST STANDARD ERROR FOR $\hat{\beta}_i$:

(i) Estimate (12.39) by OLS, which yields “se($\hat{\beta}_i$”), $\hat{\sigma}$, and the OLS residuals $\{\hat{r}_t; t = 1, \ldots, n\}$.

(ii) Compute the residuals $\{\hat{r}_t; t = 1, \ldots, n\}$ from the auxiliary regression (12.41). Then form $\hat{\alpha}_t = \hat{r}_t / \hat{\sigma}$ (for each $t$).

(iii) For your choice of $g$, compute $\hat{v}$ as in (12.42).

(iv) Compute $\text{se}(\hat{\beta}_i)$ from (12.43).

Steps in Eviews

1. **Step 1:** Obtain the original output.
   
   $$\text{inf} = 1.423 + .468\text{unem}$$

2. **Step 2:** In the output
   - Choose "Estimate" 
   - Click Options
   - Heteroskedasticity Consistent Coefficient Covariance 
   - choose Newley-West

---

### Table: Coefficient Comparison

<table>
<thead>
<tr>
<th>Variable</th>
<th>OLS</th>
<th>s.e.</th>
<th>SC s.e</th>
<th>prob</th>
<th>SC prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>1.4231</td>
<td>1.7191</td>
<td>1.5150</td>
<td>0.4118</td>
<td>0.3522</td>
</tr>
<tr>
<td>UNEM</td>
<td>0.4676</td>
<td>0.2891</td>
<td>0.2916</td>
<td>0.1125</td>
<td>0.1155</td>
</tr>
</tbody>
</table>

---

**Step 1:** Regress $\text{inf}$ on $\text{unem}$

**Step 2:** In the output
   - Choose "Estimate" 
   - Click Options
   - Heteroskedasticity Consistent Coefficient Covariance 
   - choose Newley-West

**Newey-West HAC Standard Errors & Covariance (lag truncation=3)**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.4236</td>
<td>1.7190</td>
<td>0.8281</td>
<td>0.4118</td>
</tr>
<tr>
<td>UNEM</td>
<td>0.4677</td>
<td>0.2891</td>
<td>1.6174</td>
<td>0.1125</td>
</tr>
</tbody>
</table>

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I. Properties | II. Testing | III. Remedial | IV. ARCH V. Hetero & S.C.
III. Correcting for serial correlation | IV. Serial Correlation: Quantitative Methods of Economic Analysis | Chairat Aemkulwat | 5029605 | Chairat Aemkulwat

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IV. ARCH model

- We are interested in the \textit{dynamic forms of heteroskedasticity}.

- Given a model,
  \[ y_t = \beta_0 + \beta_z z_t + u_t \]

- Suppose Gauss Markov assumptions hold.
  1. OLS have desirable properties.
  2. TS.4 implies that \( E(u_t^2 | Z) = \text{constant} \) \((Z \text{ is all n outcomes of } z_i)\)
  3. Even when variance of \( u_t \) given \( Z \) is constant, there are other ways that heteroskedasticity can arise.

OLS estimators and ARCH

- Engle (1982) suggested the conditional variance of \( u_t \) depends on past errors in the form,
  \[ E(u_t^2 | u_{t-2}, u_{t-3}, \ldots) = E(u_t^2 | u_{t-1}) = \alpha_0 + \alpha_1 u_{t-1}^2 \]

  This is called the first-order ARCH model. Note that \( \alpha_1 \neq 0 \) implies that there is dynamics in the variance equation.

- Example – UK inflation
  Engle found that a larger magnitude of the error variance in the previous time period was associated with a larger error variance in the current period.

ARCH Model

- We can write the equation in the form
  \[ u_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + v_t \]
  Let \( E(v_t | u_{t-2}, u_{t-3}, \ldots) = 0 \) and \( \alpha_1 < 1 \) for stability condition.

- Why ARCH model?
  1) It is possible to obtain consistent estimators that are asymptotically more efficient than OLS. A WLS procedure will do the trick.
  2) It gives an insight into the dynamics in the conditional variance.

Example: ARCH in Stock Return

- \text{return} = 0.180 + 0.059 \text{return}_{t-1}
  \[ [t] \quad [2.22] \quad [1.55] \]
  \[ n = 689, R^2 = .0035, R^2bar = .0020 \]

- The heteroskedasticity can be characterized by ARCH model.
  \[ \hat{u}_t^2 = 2.95 + .337 \hat{u}_{t-1}^2 \]
  \[ [t] \quad [6.766] \quad [9.37] \]

- Notes:
  1) Note that there is no serial correlation: \( \hat{\rho} = .0014 \) (t=0.038).
  2) The t-statistic (t=-9.4) is very high, indicating strong ARCH.
  3) Implications: a larger variance in stock last period implies a larger variance in stock this period.
Dependent Variable: RESID01^2
Method: Least Squares
Sample(adjusted): 4 691
Included observations: 688 after adjusting endpoints

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>2.947433</td>
<td>0.440234</td>
<td>6.695148</td>
<td>0</td>
</tr>
<tr>
<td>RESID01(-1)^2</td>
<td>0.337062</td>
<td>0.035947</td>
<td>9.376707</td>
<td>0</td>
</tr>
</tbody>
</table>

R-squared 0.113606  Mean dependent var 4.446349  Adjusted R-squared 0.112314  S.D. dependent var 11.41945  S.E. of regression 1.369665  Durbin-Watson stat 2.028071

Heteroskedasticity (ARCH model):
\[ \hat{\sigma}_t^2 = 2.95 + .337 \hat{\sigma}_{t-1}^2 \]

Steps: FGLS with hetero and AR(1) SC

Step 1: Run the regression of \( y_t \) on \( x_{1t}, x_{12}, \ldots, x_{1k} \) and obtain residuals \( \hat{u}_t \).

Step 2: Regress \( \log(\hat{u}_t^2) \) on \( x_{1t}, x_{12}, \ldots, x_{1k} \) (or \( \hat{y}_t \) and \( \hat{y}_t^2 \)) and obtain fitted values, call \( \hat{y}_t \).

Step 3: Obtain \( \hat{h}_t \) by
\[ \hat{h}_t = \exp(\hat{y}_t) \]

Step 4: Estimate the transformed equation
\[ y_t / \sqrt{\hat{h}_t} = \beta_0 / \sqrt{\hat{h}_t} + \beta_1 x_{1t} / \sqrt{\hat{h}_t} + \ldots + \beta_k x_{kt} / \sqrt{\hat{h}_t} + v_t \]

Step 5: Correct serial correlation by standard Cochrane-Orcutt or Prais-Winsten methods.

V. Heteroskedasticity and Serial Correlation

- Suppose TS.1-TS.3 hold. Consider a model
\[ y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{12} + \ldots + \beta_k x_{kt} + u_t \]
\[ u_t = \sqrt{h_t} v_t \]
\[ v_t = \rho v_{t-1} + e_t, \quad |\rho| < 1 \]

where
1) \( \{e_t\} \sim \text{i.i.d.} (0, \sigma_e^2) \)
2) \( E(e_t | X) = 0 \)
3) \( h_t = f(x_t) \)

Thus,
\[ \text{Var}(u_t) = \sigma_u^2, \quad \sigma_u^2 = \frac{\sigma_v^2}{1 - \rho^2} \]

Heteroskedasticity in Time Series Regressions

- Suppose TS.1'-TS.5' hold, the usual OLS statistics are asymptotically valid.
  - TS.3' implies that model misspecifications are ruled out.
  - TS.4' and TS.5' rule out heteroskedasticity and serial correlation.

- Serial Correlated Errors – more pressing in Time Series Heteroskedastic errors – more pressing in Cross Section.
Steps in correcting for heteroskedasticity in time series

- **Step 1**: Test for serial correlation and correct by, say, the iterative Cochrane-Orcutt method.

- **Step 2**: Test for heteroskedasticity, for example, Breusch-Pagan Test.
  
  \[ u_t^2 = \delta_0 + \delta_1 x_{1t} + \delta_2 x_{2t} + \ldots + \delta_k x_{kt} + \nu_t \]

  where the null hypothesis is \( H_0 : \delta_1 = \delta_2 = \ldots = \delta_k = 0 \)

- **Step 3**: if heteroskedasticity is found, to correct we may
  - use the heteroskedasticity-robust procedures.
  - use weighted least squares method.

Example: Heteroskedasticity in the EM Hypothesis

\[
\text{return}_t = 0.180 + 0.059\text{return}_{t-1} + \nu_t
\]

\[
\begin{array}{lll}
 t & \{2.224\} & \{1.549\} \\
 n & 689 & R^2 = 0.0035, R^2_{\text{bar}} = 0.0020 
\end{array}
\]

With such a large sample, there is no evidence against EMH. Note that there is no serial correlation; \( \rho = 0.0014 \) (t=0.038).

- EMH states that the expected return given past information should be constant. It says nothing about the conditional variance. Breusch-Pagan Test suggests

  \[
  \hat{\nu}_t^2 = 4.66 - 1.104\text{return}_{t-1}
  \]

  \[
  \begin{array}{lll}
  t & \{10.89\} & \{-5.48\} 
  \end{array}
  \]

- There is a strong evidence of heteroskedasticity.
Stock Return: No Serial Correlation:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RESID01(-1)</td>
<td>0.001405</td>
<td>0.03815</td>
<td>0.036828</td>
<td>0.9706</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.000002</td>
<td>-0.00117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>0.000002</td>
<td>2.110171</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>2.110169</td>
<td>4.332865</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum squared resid</td>
<td>3059.082</td>
<td>4.339455</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-1489.51</td>
<td>1.999909</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Breusch-Pagan Test:
\[ \hat{\mu}_t^2 = 4.66 - 1.104 \text{return}_{t-1} \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4.656501</td>
<td>0.427679</td>
<td>10.88784</td>
<td>0</td>
</tr>
<tr>
<td>RETURN(-1)</td>
<td>-1.10413</td>
<td>0.201403</td>
<td>-5.482207</td>
<td>0</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.041914</td>
<td>4.440839</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>0.040519</td>
<td>11.41207</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>11.17847</td>
<td>7.68192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>-2639.89</td>
<td>30.0546</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Durbin-Watson stat</td>
<td>1.442973</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Recap on Serial Correlation

- Properties of OLS with Serially Correlated Errors
- Testing for Serial Correlation
- Correcting for Serial Correlation
- Autoregressive conditional heteroskedasticity (ARCH) model
- Heteroskedasticity in Time Series

Good Luck! on Your Final Examination