Math 2301314 Intro to PDEs
HOMEWORK 4

<u>Due on Wednesday December 17, 2008</u>

1. Solve

$$
\begin{aligned}
u_t &= u_{xx} + \sin \pi x \quad &&; \quad 0 < x < 1,\ 0 < t < \infty \\
u(0,t) &= 0 &&; \quad 0 < t < \infty \\
u(1,t) &= 0 &&; \quad 0 < t < \infty \\
u(x,0) &= 1 &&; \quad 0 \le x \le 1.
\end{aligned}
$$

2. Consider the heat equation $u_t = k u_{\theta\theta}$ in a thin ring, and suppose that the initial temperature of the ring is $u(\theta,0) = \phi(\theta), \theta \in [0, 2\pi]$, where ϕ is a given function. Since physically θ is defined up to the addition of integer multiples of 2π, we may consider u as a 2π- periodic function of θ.

(a) The total heat energy in the ring at time t is

$$Q(t) = \int_0^{2\pi} u(\theta,t)\, d\theta .$$

Show that $Q(t)$ is a constant (independent of time).

(b) Using the separation of variables, show that the general solution of the heat equation is

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 kt} \left[A_n \cos n\theta + B_n \sin n\theta \right],$$

and find A_n, B_n in terms of ϕ.

(c) Find Q in terms of the A_n, B_n.

(d) Find $\lim_{t\to\infty} u(\theta,t)$ in terms of ϕ. Try to interpret the result physically.

3. Consider $u_t = u_{xx} + f(x)$, $0 < x < 1$, $0 < t < \infty$ with $u(x,0) = \phi(x)$ and $u_x(0,t) = 0, u_x(1,t) = 0$. Suppose that $\int_0^1 f(x)\,dx \neq 0$. Show that there are no time-independent solution. What happens to the time-dependent solution as $t \to \infty$? (Briefly explain this).