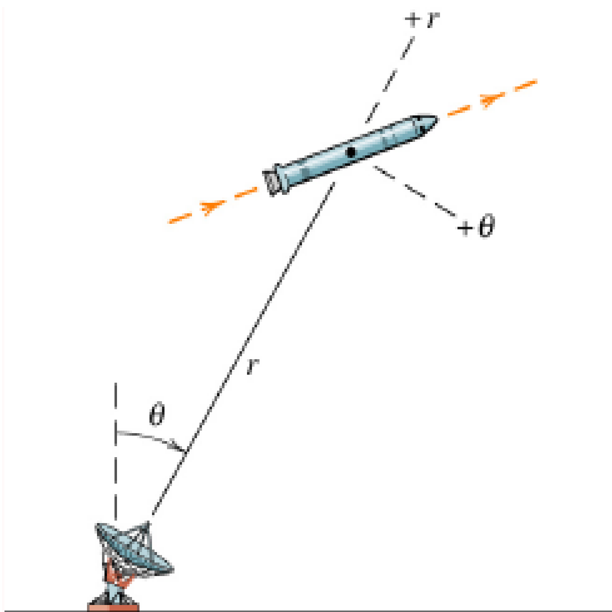
Polar Coordinates (r-θ)

3. Polar Coordinates (r-θ)

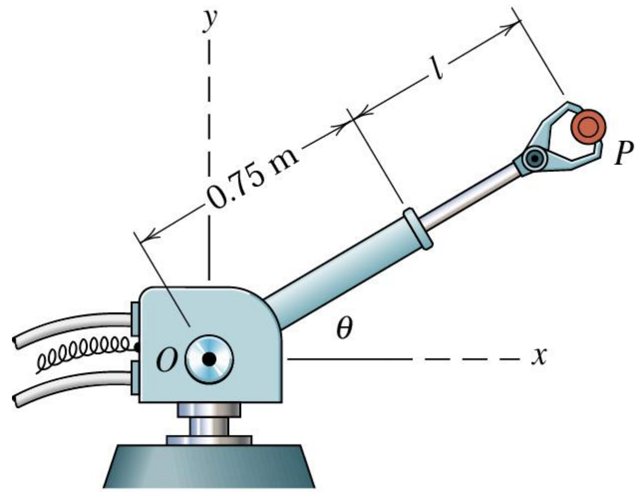
- Position
- Time derivative of unit vectors: $\frac{d\hat{e}_r}{dt}$ and $\frac{d\hat{e}_\theta}{dt}$
- Velocity
- Acceleration
- Special Case: Circular Motion
- Examples

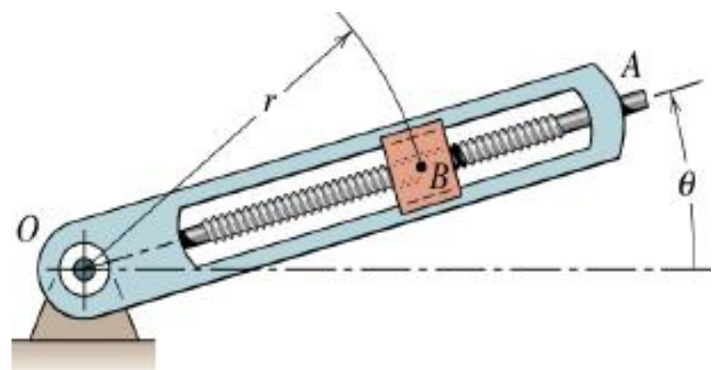
3. Polar Coordinates (r-θ)

Applications



Radar tracking

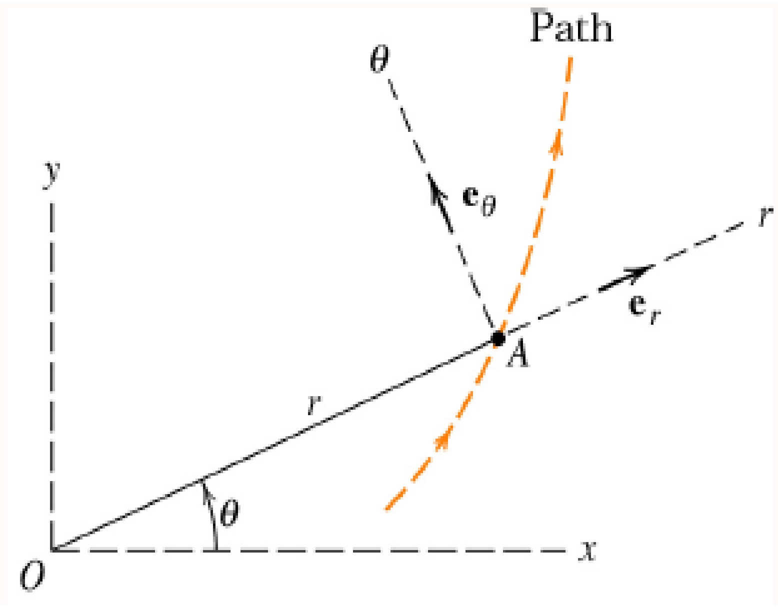




Robot Manipulator

3. Polar Coordinates (r-θ)

Position Vector



- Define a reference frame then the define r and θ
- The position vector

$$\vec{r} = r\,\hat{e}_r$$

- $\hat{e}_r$ is the unit vector in the direction of $\vec{r}$

$$\hat{e}_r = \cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}$$

- $\hat{e}_\theta$ is the unit vector in the θ direction

$$\hat{e}_\theta = -\sin(\theta)\,\hat{i} + \cos(\theta)\,\hat{j}$$

3. Polar Coordinates (*r*-*θ*)

Velocity

$$\vec{v} = \frac{d(r\hat{e}_r)}{dt} = \dot{r}\,\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

Time derivative of unit vectors

$$\hat{e}_r = \cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}$$

$$\begin{aligned}\frac{d\hat{e}_r}{dt} &= \frac{d}{dt}(\cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}) \\ &= -\sin(\theta)\dot{\theta}\,\hat{i} + \cos(\theta)\dot{\theta}\,\hat{j}\end{aligned}$$

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta$$

note that $\frac{d\hat{i}}{dt} = 0$ and $\frac{d\hat{j}}{dt} = 0$

$$\hat{e}_\theta = -\sin(\theta)\,\hat{i} + \cos(\theta)\,\hat{j}$$

3. Polar Coordinates (*r*-*θ*)

Time derivative of unit vectors

$$\hat{e}_\theta = -\sin(\theta)\,\hat{i} + \cos(\theta)\,\hat{j}$$

$$\begin{aligned}\frac{d\hat{e}_\theta}{dt} &= \frac{d}{dt}(-\sin(\theta)\,\hat{i} + \cos(\theta)\,\hat{j}) \\ &= -\cos(\theta)\dot{\theta}\,\hat{i} - \sin(\theta)\dot{\theta}\,\hat{j}\end{aligned}$$

$$\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r$$

note that $\frac{d\hat{i}}{dt} = 0$ and $\frac{d\hat{j}}{dt} = 0$

$$\hat{e}_r = \cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}$$

3. Polar Coordinates (r-θ)

Velocity

$$\vec{v} = \frac{d(r\hat{e}_r)}{dt} = \dot{r}\,\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

Velocity (Polar)

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$$

$$v = \sqrt{v_r^2 + v_\theta^2}$$

Acceleration

$$\begin{aligned}
\vec{a} &= \frac{d}{dt}(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta) \\
&= \ddot{r}\,\hat{e}_r + \dot{r}\frac{d}{dt}\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}\hat{e}_\theta \\
&= \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r
\end{aligned}$$

3. Polar Coordinates (r-θ)

Acceleration

Acceleration (Polar)

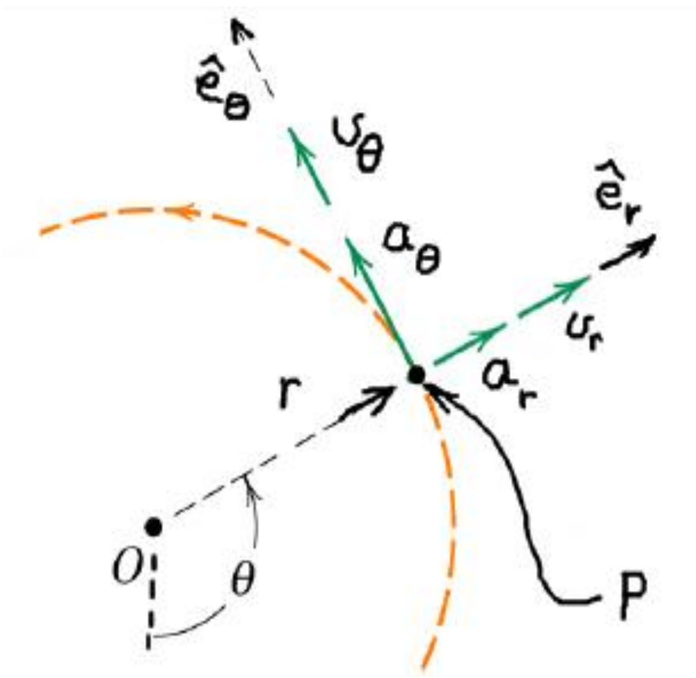
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a = \sqrt{a_r^2 + a_\theta^2}$$

3. Polar Coordinates (r-θ)

Circular Motion



- $\hat{e}_r$ points from O toward P
- $\hat{e}_\theta$ perpendicular to $\hat{e}_r$ and toward positive θ
- Velocity

$$
\begin{aligned}
v_r &= \dot{r} = 0 \\
v_\theta &= r\dot{\theta}
\end{aligned}
$$

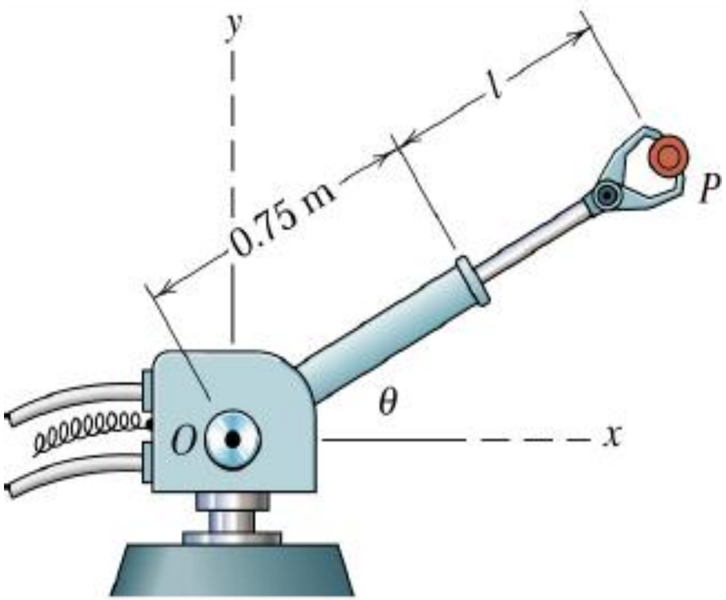
- Acceleration

$$
\begin{aligned}
a_r &= \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2 \\
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = r\ddot{\theta}
\end{aligned}
$$

3. Polar Coordinates (*r*-*θ*)

Example 1: Robot Arm

The robot arm is elevating and extending simultaneously. At a given instant, $\theta = 30°$, $\dot{\theta} = 10$ deg/s constant, $l = 0.5$ m, $\dot{l} = 0.2$ m/s, and $\ddot{l} = -0.3$ m/s^2. Compute the magnitude of the velocity, $\vec{v}$, and acceleration, $\vec{a}$, of the gripped part P. In addition, express $\vec{v}$ in terms of the unit vectors $\hat{i}$ and $\hat{j}$.



Ans:

$$v_r = 0.2 \text{ m/s} \qquad a_r = -0.338$$

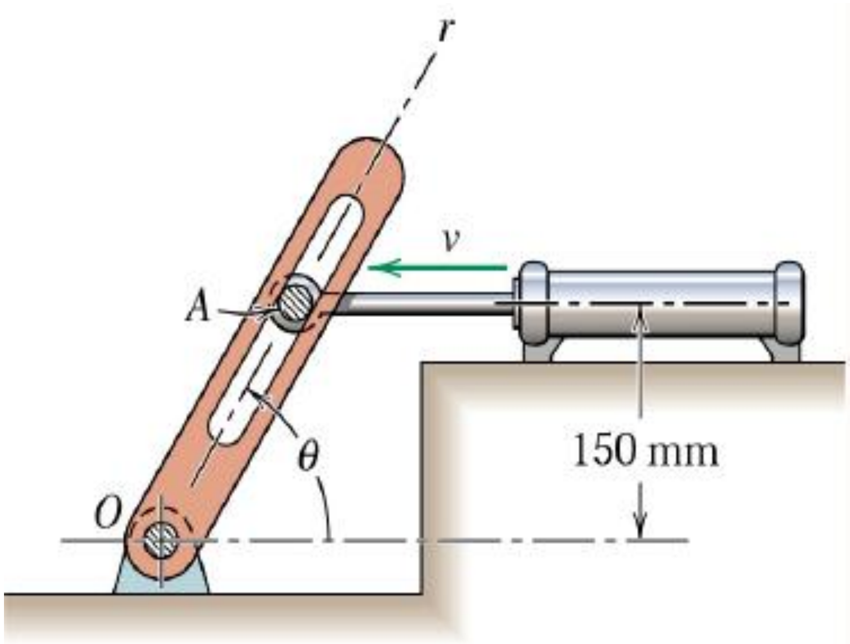
$$v_\theta = 0.218 \text{ m/s} \qquad a_\theta = 0.07$$

$$\vec{v} = 0.064\hat{i} + 0.289\hat{j} \text{m/s}$$

3. Polar Coordinates (r-θ)

Example 2: Hydraulic Cylinder

The piston of the hydraulic cylinder gives pin A a constant velocity v = 1.5 m/s in the direction shown for an interval of its motion. For the instant when θ = 60°, determine $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$, where $r = \overline{OA}$

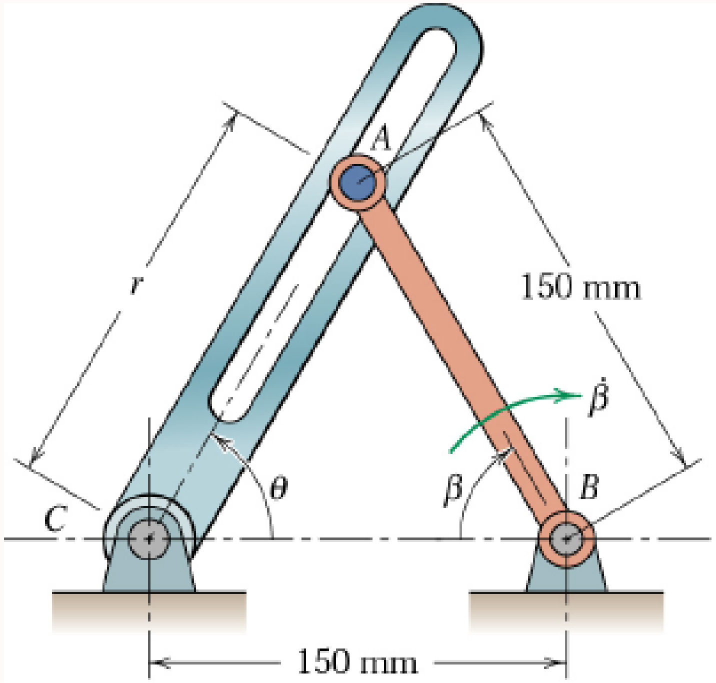


Ans: -0.75 m/s, 7.5 rad/s, 9.74 m/s^2, 65 rad/s^2

3. Polar Coordinates (r-θ)

Example 3: Two links

Link AB rotates through a limited range of the angle β, and its end A causes the slotted link AC to rotate also. For the instant represented where β = 60° and $\dot{\beta}$ = 0.6 rad/s constant, determine the corresponding values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$.



Ans: $\dot{r} = 0.078\,\text{m/s}$ $\dot{\theta} = -0.3\,\text{rad/s}$

$\ddot{r}$ = -0.0135 m/s^2 and $\ddot{\theta}$ = 0