Chapter 3
Response of SDOF Systems due to Harmonic Excitation

- Introduction

3.1 Harmonic Excitations
- Undamped
- Viscously Damped
- Frequency Response Functions
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The external force can be supplied through either an applied force or an imposed displacement excitation, which may be harmonic, nonharmonic but periodic, nonperiodic (arbitrary), or random in nature.
Introduction

- *Forced vibrations* occurs when external energy is supplied to the system during vibration.

- *Harmonic response* results when the system responds to a harmonic excitation.

- *Transient response* is the response of a dynamic system to initial conditions. This response will die out over a long period of time. [similar to homogeneous solution]

- *Steady State response* is the response of a dynamic system to harmonic or periodic excitations. This response will exist as long as there is an excitation. [similar to particular solution]

- *Total response* = transient + steady state responses
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Consider an undamped system subjected to a harmonic force. If a force \( F(t) = F_0 \cos(\omega t) \) acts on the mass \( m \) of the system,

\[
\dot{m}\ddot{x}(t) + kx(t) = F_0 \cos(\omega t)
\]

The homogeneous solution is given by:

\[
x_h(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t
\]

where \( \omega_n = \sqrt{\frac{k}{m}} \) is the natural frequency.
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Because the exciting force and particular solution is harmonic and has same frequency, we can assume a solution in the form:

$$x_p(t) = X \cos \omega t$$

where $X$ is the max amplitude of $x_p(t)$

Solve for $X$ by substituting it into EOM, we have

$$X = \frac{F_0}{k - m\omega^2}$$

Thus,

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Using initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, 

$$A_1 = x_0 - \frac{F_0}{k - m\omega^2}, \quad A_2 = \frac{\dot{x}_0}{\omega_n}$$

Hence, 

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t$$

$$+ \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \quad \text{Note: } \omega \neq \omega_n$$

Example of a total response
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Question: What is the total response if \( F(t) = F_0 \sin(\omega t) \) ?

Warning: you cannot switch sin and cos in the response shown in the previous page. Need to redo the whole problem.
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Animation: Force applied to the mass of a damped SDOF oscillator on a rigid foundation

Three identical damped 1-DOF mass-spring oscillators, all with natural frequency $f_n = 1\ \text{Hz} \ (\omega_n = 2\pi)$, are initially at rest. A time harmonic force $F = F_0 \cos(2\pi f t)$ is applied to each of three damped 1-DOF mass-spring oscillators starting at time $t=0$. The driving frequencies $\omega$ of the applied forces are (matching colors) $f = 0.4\ \text{Hz}$, $f = 1.01\ \text{Hz}$, $f = 1.6\ \text{Hz}$

http://www.kettering.edu/~drussell/Demos/SHO/mass-force.html
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 1. When \( 0 < \omega / \omega_n < 1 \), the denominator of \( X \) is positive. Thus, \( X \) is positive. The harmonic (particular) response of the system is in phase with external force, shown in figure.

\[
F(t) = F_0 \cos \omega t
\]

\[
x_p(t) = X \cos \omega t
\]

\[
X = \frac{F_0}{k - m \omega^2}
\]
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 2. When $\omega / \omega_n = 1$, the amplitude $X$ becomes infinite. This condition, for which the forcing frequency is equal to the natural frequency of the system, is called resonance. Hence, the total response if the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$+ \frac{F_0 \omega_n t}{2k} \sin \omega_n t$$
3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 3. When $\omega / \omega_n > 1$, the denominator is negative. Thus, $X$ is negative. The harmonic response of the system is 180 degrees out of phase with external force, shown in figure.

$F(t) = F_0 \cos \omega t$

![Graph showing the harmonic response of the system](image-url)
3.1 Harmonic Excitations

Summary:

Undamped response = homogeneous + particular response

if $F(t) = F_0 \cos(\omega t)$ then $x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t$

if $F(t) = F_0 \sin(\omega t)$ then $x(t) = x_0 \cos \omega_n t + \frac{1}{\omega_n} \left( \dot{x}_0 - \frac{F_0\omega}{k - m\omega^2} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \sin \omega t$

<table>
<thead>
<tr>
<th>Response</th>
<th>homogeneous</th>
<th>particular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>$\omega_n$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>Amplitude</td>
<td>depends on IC and force input (sin or cos)</td>
<td>Fixed</td>
</tr>
</tbody>
</table>
3.1 Harmonic Excitations

Example: Plate Supporting a Pump

A reciprocating pump, having a mass of 68 kg, is mounted at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm, clamped along two edges as shown in Figure. During operation of the pump, the plate is subjected to a harmonic force, \( F(t) = 220 \cos 62.832t \) N. Find the amplitude of vibration of the plate. Young’s modulus (E) = 200 GPa. The bending stiffness of the beam is given by

\[
k = \frac{192EI}{l^3}
\]
3.1 Harmonic Excitations

Solution

The area moment of inertia $I$ is

$$I = \frac{1}{12} \text{width} \times \text{thickness}^3$$

$$= \frac{1}{12} (50 \times 10^{-2})(10^{-2})^3 = 4.17 \times 10^{-9} \text{ m}^4$$

Stiffness:

$$k = \frac{192EI}{l^3} = \frac{(192)(200 \times 10^9)(41.67 \times 10^{-9})}{(250 \times 10^{-2})^3} = 102400 \text{ N/m}$$

Hence, the amplitude

$$|X| = \left| \frac{F_0}{k - m\omega^2} \right| = \left| \frac{220}{102400 - 68(62.83)^2} \right|$$

$$= 0.001325 \text{ m} = 1.325 \text{ mm}$$