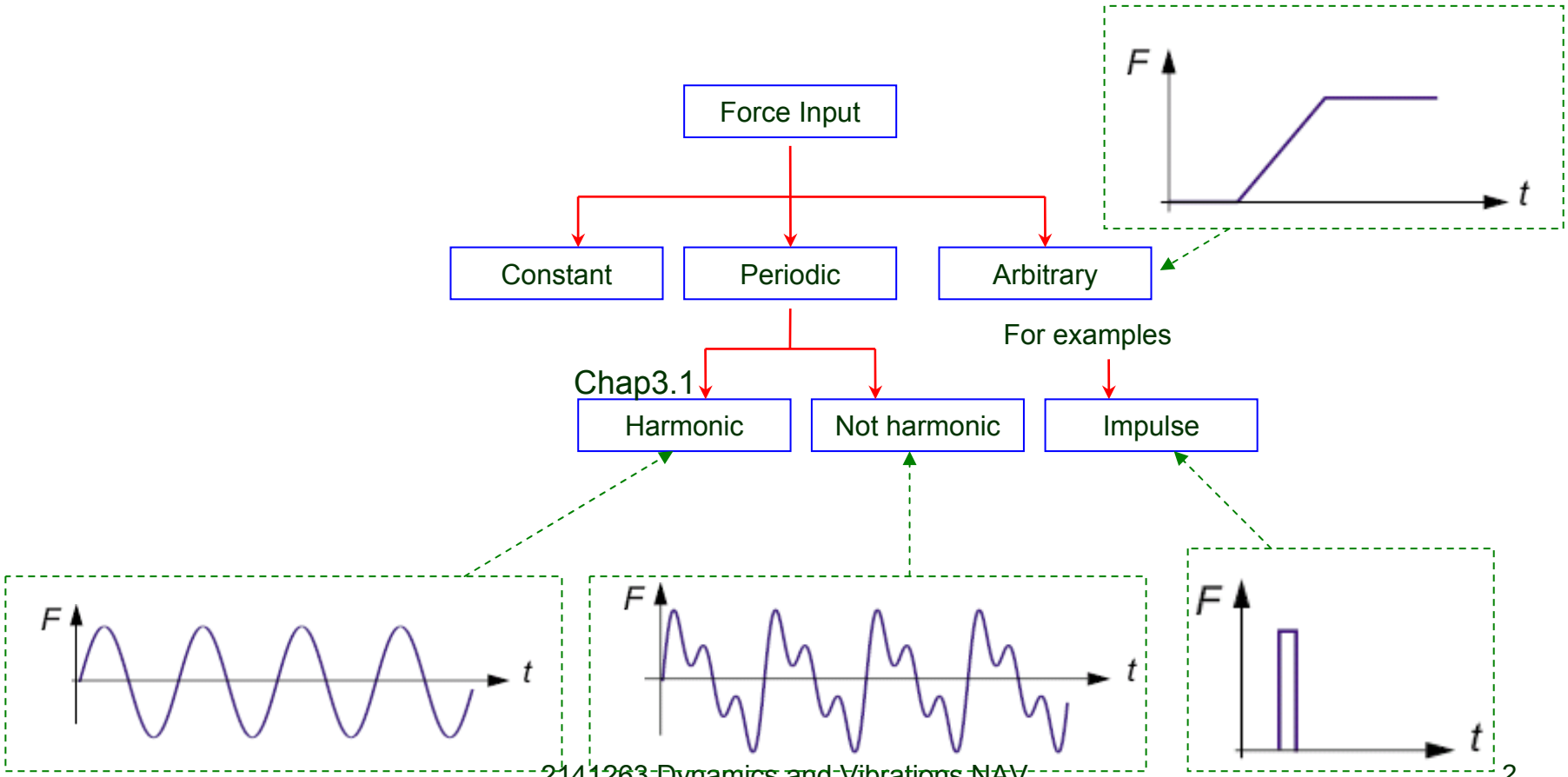
Chapter 3
Response of SDOF Systems due to Harmonic Excitation

- Introduction
- 3.1 Harmonic Excitations
  - Undamped
  - Viscously Damped
  - Frequency Response Functions
  - Frequency Response Plots
- 3.2 Applications

Introduction

- The external force can be supplied through either an applied force or an imposed displacement excitation, which may be harmonic, nonharmonic but periodic, nonperiodic (arbitrary), or random in nature.



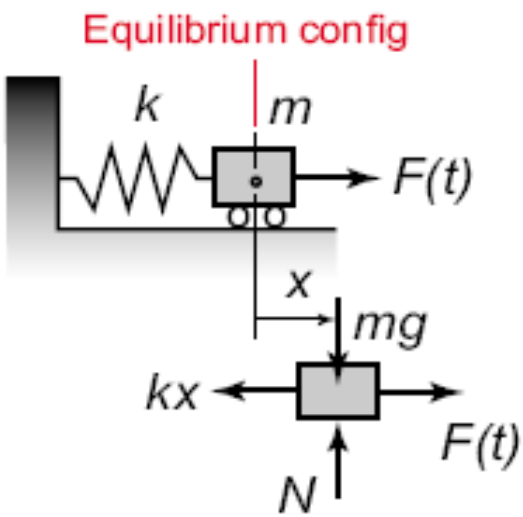
Introduction

- *Forced vibrations* occurs when external energy is supplied to the system during vibration
- *Harmonic response* results when the system responses to a harmonic excitation
- *Transient response* is the response of a dynamic system to initial conditions. This response will die out over a long period of time. [similar to homogeneous solution]
- *Steady State response* is the response of a dynamic system to harmonic or periodic excitations. This response will exist as long as there is an excitation. [similar to particular solution]
- *Total response = transient + steady state responses*

3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Consider an undamped system subjected to a harmonic force. If a force $F(t) = F_0 \cos(\omega t)$ acts on the mass *m* of the system,



$$m\ddot{x}(t) + kx(t) = F_0 \cos(\omega t)$$

The homogeneous solution is given by:

$$x_h(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

where $\omega_n = \sqrt{\dfrac{k}{m}}$ is the natural frequency.

3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Because the exciting force and particular solution is harmonic and has same frequency, we can assume a solution in the form:

$$x_p(t) = X \cos \omega t$$

where X is the max amplitude of $x_p(t)$

Solve for X by substituting it into EOM, we have

$$X = \frac{F_0}{k - m\omega^2}$$

Thus,

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Using initial conditions $x(t=0)=x_0$ and $\dot{x}(t=0)=\dot{x}_0$,

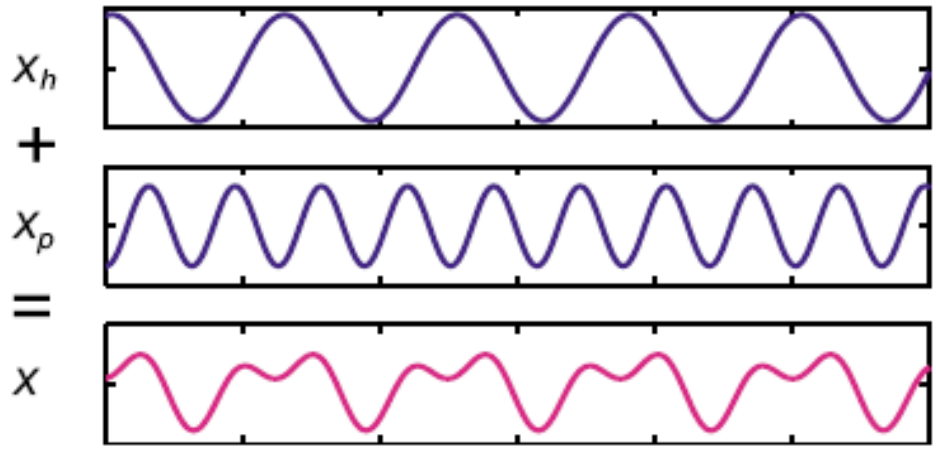
$$A_1 = x_0 - \frac{F_0}{k-m\omega^2}, \qquad A_2 = \frac{\dot{x}_0}{\omega_n}$$

Hence,

$$x(t) = \left(x_0 - \frac{F_0}{k-m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k-m\omega^2}\right)\cos\omega t$$

Note: $\omega \neq \omega_n$

Example of a total response



3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Question: What is the total response if $F(t) = F_0 \sin(\omega t)$?

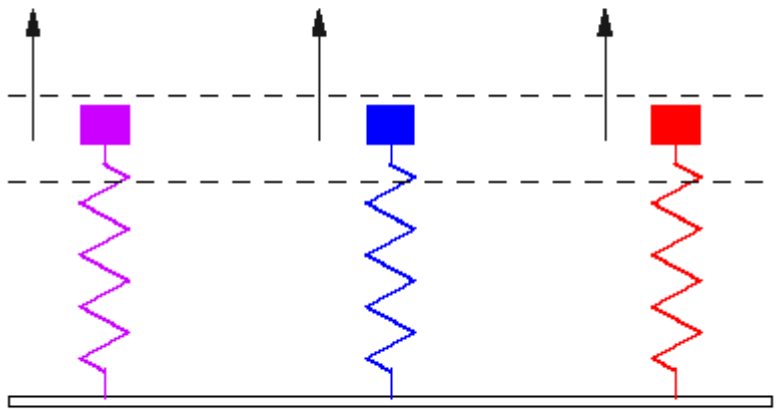
Warning: you cannot switch sin and cos in the response shown in the previous page. Need to redo the whole problem.

3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Animation: Force applied to the mass of a damped SDOF oscillator on a rigid foundation

Three identical damped 1-DOF mass-spring oscillators, all with natural frequency **f_n = 1 Hz (ω_n= 2¶)**, are initially at rest. A time harmonic force $F = F_0 \cos(2 ¶ f t)$ is applied to each of three damped 1-DOF mass-spring oscillators starting at time t=0. The driving frequencies ω of the applied forces are (matching colors) **f = 0.4 Hz**, **f = 1.01 Hz**, **f = 1.6 Hz**

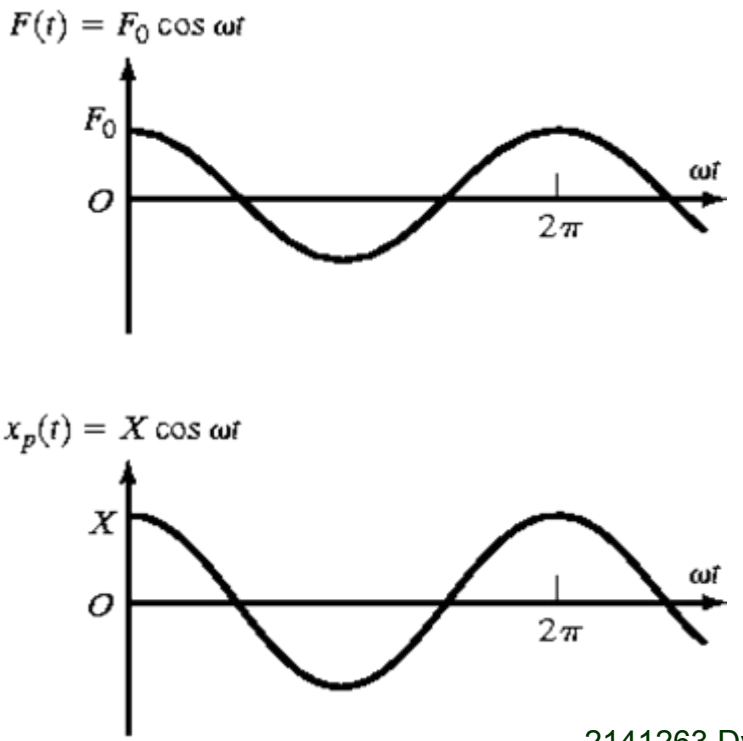


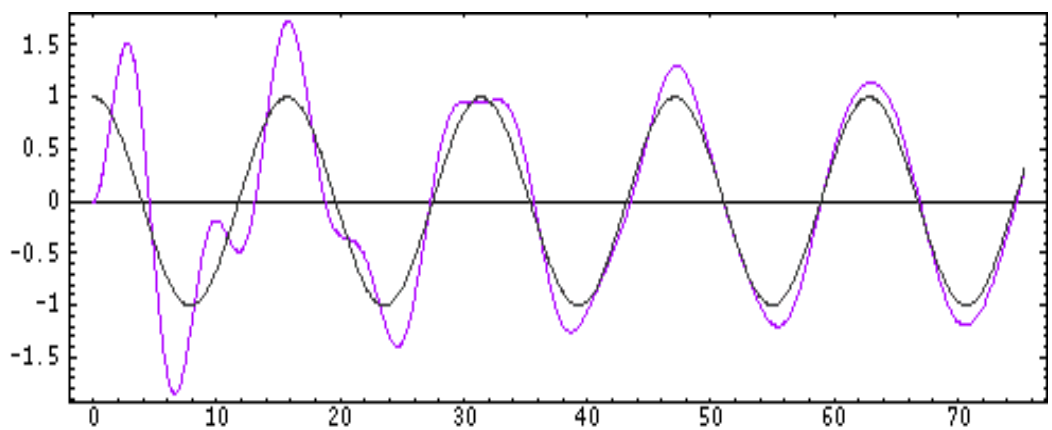
http://www.kettering.edu/~drussell/Demos/SHO/mass-force.html

3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 1. When $0 < \omega / \omega_n < 1$, the denominator of $X = \dfrac{F_0}{k - m\omega^2}$ is positive. Thus, X is positive. The harmonic (particular) response of the system is in phase with external force, shown in figure.



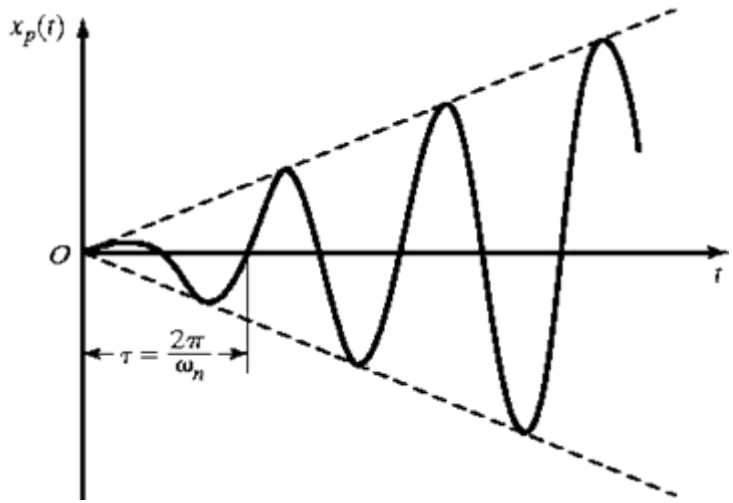


3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 2. When ω/ω_n = 1, the amplitude X becomes infinite. This condition, for which the forcing frequency is equal to the natural frequency of the system, is called *resonance*. Hence, the total response if the system at resonance becomes

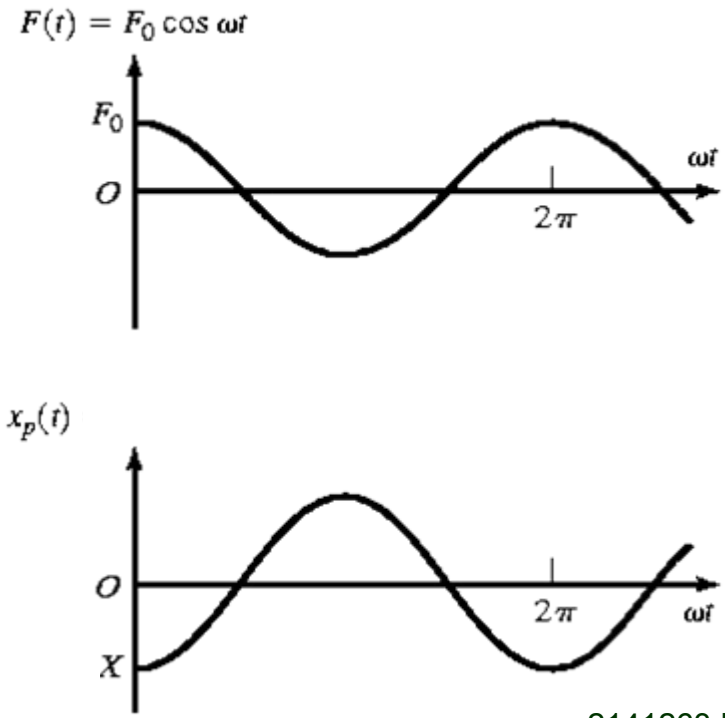
$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0 \omega_n t}{2k} \sin \omega_n t$$

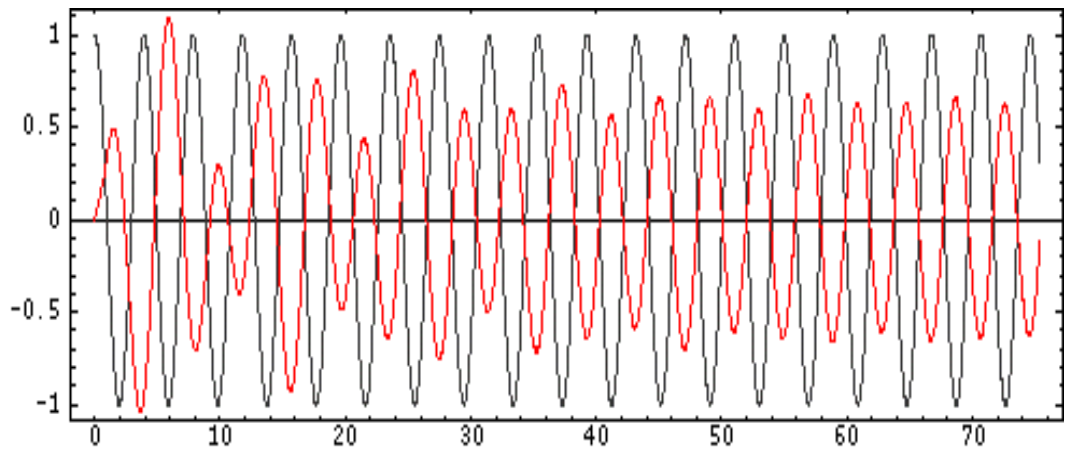


3.1 Harmonic Excitations

3.1.1 Undamped SDOF

Case 3. When ω / ω_n > 1, the denominator is negative. Thus, X is negative. The harmonic response of the system is 180 degrees out of phase with external force, shown in figure.





3.1 Harmonic Excitations

Summary:

Undamped response = homogeneous + particular response

if $F(t) = F_0 \cos(\omega t)$ then $x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t$

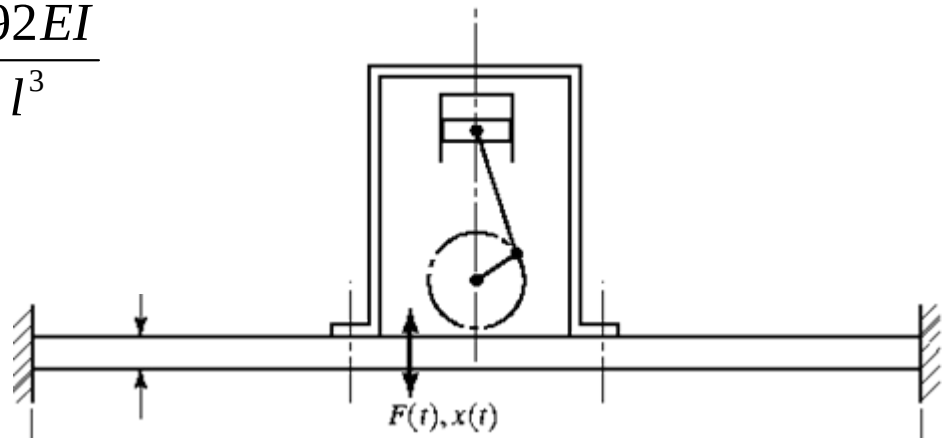
if $F(t) = F_0 \sin(\omega t)$ then $x(t) = x_0 \cos \omega_n t + \frac{1}{\omega_n} \left(\dot{x}_0 - \frac{F_0 \omega}{k - m\omega^2} \right) \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \sin \omega t$

Response	homogeneous	particular
Frequency	ω_n	ω
Amplitude	depends on IC and force input (sin or cos)	Fixed

3.1 Harmonic Excitations

Example: Plate Supporting a Pump

A reciprocating pump, having a mass of 68 kg, is mounted at the middle of a steel plate of thickness 1cm, width 50 cm, and length 250 cm, clamped along two edges as shown in Figure. During operation of the pump, the plate is subjected to a harmonic force, F(t) = 220 cos62.832t N. Find the amplitude of vibration of the plate. Young's modulus (E) = 200 GPa. The bending stiffness of the beam is given by $k = \frac{192EI}{l^3}$



3.1 Harmonic Excitations

Solution

The area moment of inertia *I* is

$$I = \frac{1}{12} width \times thickness^3$$

$$= \frac{1}{12}(50\times10^{-2})(10^{-2})^3 = 41.67\times10^{-9}\ \mathrm{m}^4$$

Stiffness:

$$k = \frac{192EI}{l^3} = \frac{(192)(200\times10^9)(41.67\times10^{-9})}{(250\times10^{-2})^3} = 102400\ \mathrm{N/m}$$

Hence, the amplitude

$$|X| = \left|\frac{F_0}{k - m\omega^2}\right| = \left|\frac{220}{102400 - 68(62.83)^2}\right|$$

$$= 0.001325\ \mathrm{m} = 1.325\ \mathrm{mm}$$