TECHNOLOGY

Specification of Technology

- Input and Output can be specified in arbitrarily fine detail

- n possible goods can be used as input and/or output

- Net Output : $y_j = y_j^{output} - y_j^{input}$

$y_j > 0 \rightarrow$ net output, $y_j < 0 \rightarrow$net input

- Production Plan = a list of net outputs, represented by a vector $\mathbf{y}$ in R^n

- A Production Possibilities Set Y= a set of all technologically feasible production plans

- Restictions on production plans : e.g. a vector $\mathbf{z}$ in R^n could be a list of a maximum amount of inputs and outputs that can be produced in a given time period.

- Restricted or Short-run production possibilities set $Y(\mathbf{z}) \subset Y$, e.g. factor n fixed at $\overline{y}_n \rightarrow Y(\overline{y}_n) = \{\mathbf{y}$ in Y : $y_n = \overline{y}_n\}$

Input Requirement Set :

$V(y) = \{\mathbf{x}$ in R^n_+ : $(y, -\mathbf{x})$ is in $Y\}$
is the set of all input bundles $\mathbf{x}$ that produces at least y unit of output
Note: only one output y, net output is written as $(y, -\mathbf{x})$

Isoquant :

$Q(y) = \{\mathbf{x}$ in R^n_+ : $\mathbf{x}$ is in $V(y)$ and $\mathbf{x}$ is not in $V(y')$ for $y' > y\}$
gives all input bundles that produces exactly y units of output

Production Function :

$f(\mathbf{x}) = \{y$ in R : y is the maximum output associated with $-\mathbf{x}$ in $Y\}$

Technologically Efficient production plans :

$\mathbf{y}$ is technologically efficient if there is no $\mathbf{y}'$ in Y such that $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$

Transformation Function :

$T : R^n \rightarrow R$ where $T(\mathbf{y}) = 0$ iff y is efficient
can be used to describe a set of technologically efficient production plans.
It is an n-dimensional analog of a production function.

Example: **Cobb-Douglas Technology**

Let $0 < a < 1$
$Y = \{(y, -x_1, -x_2)$ in R^3: $y \leq x_1^a x_2^{1-a}\}$

$V(y) = \{(x_1, x_2)$ in R^2_+: $y \leq x_1^a x_2^{1-a}\}$
$Q(y) = \{(x_1, x_2)$ in R^2_+: $y = x_1^a x_2^{1-a}\}$
$Y(z) = \{(y, -x_1, -x_2)$ in R^3: $y \leq x_1^a x_2^{1-a}, x_2 = z\}$
$T(y, x_1, x_2) = y - x_1^a x_2^{1-a}$
$f(x_1, x_2) = x_1^a x_2^{1-a}$

Example: **Leontief Technology**

Let $a > 0$ and $b > 0$
$Y = \{(y, -x_1, -x_2)$ in R^3: $y \leq \min(ax_1, bx_2)\}$
$V(y) = \{(x_1, x_2)$ in R^2_+: $y \leq \min(ax_1, bx_2)\}$
$Q(y) = \{(x_1, x_2)$ in R^2_+: $y = \min(ax_1, bx_2)\}$
$T(y, x_1, x_2) = y - \min(ax_1, bx_2)$
$f(x_1, x_2) = \min(ax_1, bx_2)$

Monotonic Technologies

Under free disposal, if $\mathbf{x}$ is a feasible way to produce y and $\mathbf{x}' \geq \mathbf{x}$ should also be a feasible way to produce y

Monotonicity

For input requirement sets: If $\mathbf{x}$ is in $V(y)$ and $\mathbf{x}' \geq \mathbf{x}$, then $\mathbf{x}'$ is in $V(y)$

For production sets: If $\mathbf{y}$ is in Y and $\mathbf{y}' \leq \mathbf{y}$, then $\mathbf{y}'$ is in Y

$\mathbf{y}'$ produces an equal or samller amount of all outputs by using at least as much of all inputs compared to $\mathbf{y}$

Convex Technologies

Convexity

For input requirement sets: If $\mathbf{x}$ and $\mathbf{x}'$ are in $V(y)$, then $t\mathbf{x} + (1-t)\mathbf{x}'$ is in $V(y)$ for all $0 \leq t \leq 1$

For production sets: If $\mathbf{y}$ and $\mathbf{y}'$ are in Y, then $t\mathbf{y} + (1-t)\mathbf{y}'$ is in Y for all $0 \leq t \leq 1$

Convex production set implies convex input requirement set

Proof. If Y is a convex set and $(y, -\mathbf{x})$ and $(y, -\mathbf{x}')$ are in Y, then $(ty + (1-t)y, -t\mathbf{x} - (1-t)\mathbf{x}')$ must be in Y, i.e. $(y, -t\mathbf{x} - (1-t)\mathbf{x}')$ is in Y

Hence, if $\mathbf{x}$ and $\mathbf{x}'$ are in $V(y)$, then $t\mathbf{x} + (1-t)\mathbf{x}'$ is in $V(y)$.

Convex input requirement set is equivalent to quasiconcave production function

Proof. By definition, a function is quasiconcave iff it has a convex upper contour set. Since $V(y) = \{\mathbf{x} : f(\mathbf{x}) \geq y\}$ is the upper contour set of $f(\mathbf{x})$, then if $V(y)$ is convex, $f(\mathbf{x})$ is quasiconcave.

Regular Technologies

Regular

$V(y)$ is a closed and nonempty set for all $y \geq 0$

Nonempty $\longrightarrow$ there is a way to produce any given amount of output y

Closed $\longrightarrow$ $V(y)$ includes its own boundary. If a sequence of input bundles that can produce y converges to an input $\mathbf{x}^0$ then $\mathbf{x}^0$ must be capable of producing y

The Technical Rate of Substitution

The technical rate of substitution TRS measures how one of the inputs must adjust in order to keep output constant when another input changes. For 2-dimensional case, TRS measures the slope of the isoquant

TRS of x_2 for $x_1 = \frac{dx_2}{dx_1} = \frac{\partial f(\mathbf{x})/\partial x_1}{\partial f(\mathbf{x})/\partial x_2}$

The Elasticity of Substitution

The elasticity of substitution measures the % change in the factor ratio divided by the % change in the TRS. The elasticity of substitution measures the curvature of the isoquant.

Elasticity of substitution $\sigma = \frac{\triangle(x_2/x_1)}{x_2/x_1} / \frac{\triangle \text{TRS}}{\text{TRS}}$

For small change, $\sigma = \frac{d(x_2/x_1)}{x_2/x_1} / \frac{d\text{TRS}}{\text{TRS}} = \frac{\text{TRS}}{x_2/x_1} \frac{d(x_2/x_1)}{d\text{TRS}}$

Homogeneity

For any scalar k, a function $f(x_1, ...x_2)$ is homogenoeus of degree k if
$f(tx_1, ...tx_2) = t^k f(x_1, ...x_2)$ for all $x_1, ...x_2$ and all $t > 0$

Returns to Scale

If we scale all inputs up by some amount $t \geq 0$, what will happen to output?

Constant Returns to Scale

A technology exhibits constant returns to scale if one of the following is satisfied

i) $\mathbf{y}$ is in Y implies $t\mathbf{y}$ is in Y, for all $t \geq 0$

ii) $\mathbf{x}$ is in $V(y)$ implies $t\mathbf{x}$ is in $V(y)$, for all $t \geq 0$

iii) $f(tx) = tf(x)$, for all $t \geq 0$, i.e. the production function is homogeneous of degree 1

Increasing Returns to Scale

A technology exhibits increasing returns to scale if $f(tx) > tf(x)$, for all $t > 1$

Decreasing Returns to Scale

A technology exhibits decreasing returns to scale if $f(tx) < tf(x)$, for all $t > 1$

The above measures are global measures. The following gives a local measure of returns to scale

The Elasticity of Scale

The elasticity of scale measures % increase in output due to a one % increase in all inputs

Elasticity of scale $e(x) = \frac{df(tx)}{f(tx)} / \frac{dt}{t}$ evaluated at $t = 1$

or $e(x) = \frac{df(tx)}{dt} \frac{t}{f(tx)}$evaluated at $t = 1$

The technology exhibits locally IRS if $e(x) > 1$, CRS if $e(x) = 1$, and DRS if $e(x) < 1$.

Monotonic Transformation

A function $g : R \rightarrow R$ is said to be (positive) monotonic transformation if g is a strictly increasing function
If $x > y$ then $g(x) > g(y)$

Homothetic Function

A homothetic function is a monotonic transformation of a homogeneous function (of degree 1)
i.e. $f(x)$ is homothetic iff it can be written as $f(x) = g(h(x))$ where $h(\cdot)$ is homogeneous and $g(\cdot)$ is a monotonic transformation

Homogeneity is a cardinal property whereas Homotheticity is an ordinal property.