Introduction to Optimization Problem with Equality Constraint

$$\text{Maximise or Minimize } z(x_1, x_2, ...x_n)$$

$$\begin{aligned}
\text{subject to } g_1(x_1, x_2, ...x_n) &= a_1 \\
g_2(x_1, x_2, ...x_n) &= a_2 \\
&. \\
&. \\
g_m(x_1, x_2, ...x_n) &= a_m
\end{aligned}$$

Let $\mathbf{x} = (x_1, x_2, ...x_n)$ and $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), ...g_m(\mathbf{x}))^t$ and $\mathbf{a} = (a_1, a_2, ...a_m)^t$

We can write the above problem as

$$\text{Maximise or Minimize } z(\mathbf{x})$$

$$\text{subject to } \mathbf{g}(\mathbf{x}) = \mathbf{a}$$

Letting $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ...\lambda_m)$.
We can set up the Lagrangian as

$$L(\mathbf{x}, \boldsymbol{\lambda}) = z(\mathbf{x}) - \lambda_1 \left[g_1(\mathbf{x}) - a_1\right] - .... - \lambda_m \left[g_m(\mathbf{x}) - a_m\right]$$

or

$$L(\mathbf{x}, \boldsymbol{\lambda}) = z(\mathbf{x}) - \boldsymbol{\lambda} \left[\mathbf{g}(\mathbf{x}) - \mathbf{a}\right]$$

First-order conditions

If $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is the solution to the above problem, we must have

$$
\begin{aligned}
\frac{\partial L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_1} &= 0 \\
&\;. \\
&\;. \\
\frac{\partial L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_n} &= 0 \\
&\text{and} \\
\frac{\partial L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial \lambda_1} &= 0 \\
&\;. \\
&\;. \\
\frac{\partial L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial \lambda_m} &= 0
\end{aligned}
$$

Second-order conditions

1) For maximization problem:

Formally, the Hessian of L w.r.t. $\mathbf{x}$ at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, denoted by $D^2_{\mathbf{x}}L(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, is negative definite subject to the constraint $D\mathbf{g}(\mathbf{x}^*)\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$

In other word, $\mathbf{h}^T D^2_{\mathbf{x}}L(\mathbf{x}^*, \boldsymbol{\lambda}^*)\mathbf{h} < 0$ subject to the constraint $D\mathbf{g}(\mathbf{x}^*)\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$

2) For minimization problem:

Formally, the Hessian of L w.r.t. $\mathbf{x}$ at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, denoted by $D^2_{\mathbf{x}}L(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, is positive definite subject to the constraint $D\mathbf{g}(\mathbf{x}^*)\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$

In other word, $\mathbf{h}^T D^2_{\mathbf{x}}L(\mathbf{x}^*, \boldsymbol{\lambda}^*)\mathbf{h} > 0$ subject to the constraint $D\mathbf{g}(\mathbf{x}^*)\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$

Note:

$$D^2_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \begin{pmatrix} \frac{\partial^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_1^2} & . & . & \frac{\partial^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_n \partial x_1} \\ . & . & & . \\ . & & . & . \\ \frac{\partial^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_1 \partial x_n} & . & . & \frac{\partial^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_n^2} \end{pmatrix}$$

and $D\mathbf{g}(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial g_1(\mathbf{x}^*)}{\partial x_1} & . & . & \frac{\partial g_1(\mathbf{x}^*)}{\partial x_n} \\ . & . & & . \\ . & & . & . \\ \frac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & . & . & \frac{\partial g_m(\mathbf{x}^*)}{\partial x_n} \end{pmatrix}$

Practical test for definiteness of matrix

First, construct the Bordered Hessian, denoted by

$$D^2_{(\mathbf{x},\boldsymbol{\lambda})}L(\mathbf{x}^*,\boldsymbol{\lambda}^*) = \begin{pmatrix} \mathbf{0} & -D\mathbf{g}(\mathbf{x}^*) \\ -D\mathbf{g}(\mathbf{x}^*)^T & D^2_{\mathbf{x}}L(\mathbf{x}^*,\boldsymbol{\lambda}^*) \end{pmatrix}$$

where $\mathbf{0}$ is an $m \times m$ zero matrix

1) For maximization problem:

The bordered-preserving principal minor of order k has the sign $(-1)^k$ for $k = 2, 3, ..., n$

2) For minimization problem:

The bordered-preserving principal minors of order k are all negative for $k = 2, 3, ..., n$

Note: The bordered-preserving principal minor of order 1 is the determinant of

$$\begin{pmatrix} 0 & \cdot & \cdot & 0 & -\frac{\partial g_1(\mathbf{x}^*)}{\partial x_1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & -\frac{\partial g_m(\mathbf{x}^*)}{\partial x_1} \\ -\frac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & \cdot & \cdot & -\frac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & \frac{\partial^2 L(\mathbf{x}^*,\boldsymbol{\lambda}^*)}{\partial x_1^2} \end{pmatrix}$$

COST MINIMIZATION

Calculus for Cost Minimization

Consider a problem of finding a cost-minmizing way to produce a given level of output

$$\min_{\mathbf{x}} \mathbf{wx}$$

$$\text{subject to } f(\mathbf{x}) = y$$

The Lagrangian is

$$L(\mathbf{x}, \lambda) = \mathbf{wx} - \lambda \left[f(\mathbf{x}) - y \right]$$

<u>F.O.C.'s are</u>

$$\begin{aligned} w_i - \lambda \frac{\partial f(\mathbf{x}^*)}{\partial x_i} &= 0, \text{ for } i = 1, 2, \ldots.n \\ f(\mathbf{x}^*) - y &= 0 \end{aligned}$$

Interpretation of F.O.C.'s

$$\frac{w_i}{w_j} = \frac{\frac{\partial f(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial f(\mathbf{x}^*)}{\partial x_j}} = RTS_j \text{ for } i$$

$RTS_{j \text{ for } i}$ or the rate of technical substitution of j for i is the rate at which factor j can be substituted for factor i holding output level constant.
$\frac{w_i}{w_j}$ is the economic rate of substitution, the rate at which factor j can be substituted for factor i holding cost constant.

<u>S.O.C.'s are</u>

Suppose there are two factor inputs x_1 and x_2

$$L(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 - \lambda \left[f(x_1, x_2) - y \right]$$

$$D^2_{(x_1, x_2, \lambda)} L(x_1^*, x_2^*, \lambda^*) = \begin{pmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} \end{pmatrix}$$

Hence, to satisfy S.O.C.,
the bordered-preserving principal minor of order 2

$$= \begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{12} \\ -f_2 & -\lambda f_{21} & -\lambda f_{22} \end{vmatrix} < 0$$

Graphical Representation

Definition: **Conditional Factor Demand Function**

$\mathbf{x}(\mathbf{w}, y)$ is the optimal choice of $\mathbf{x}$ that minimizes the cost of producing y given $\mathbf{w}$.

Definition: **Cost Function**

$c(\mathbf{w}, y) = \mathbf{wx}(\mathbf{w}, y)$ is the minimum cost at factor prices $\mathbf{w}$ and output level y.

Difficulties

1) The technology may not be a differentiable function

2) The above F.O.C. conditions are valid only for interior solution, i.e. for $x_i > 0$ for all $i = 1, 2...n$

Otherwise,

$$\begin{aligned} w_i - \lambda \frac{\partial f(\mathbf{x}^*)}{\partial x_i} &\geq 0 \text{ if } x_i^* = 0 \\ w_i - \lambda \frac{\partial f(\mathbf{x}^*)}{\partial x_i} &= 0 \text{ if } x_i^* > 0 \end{aligned}$$

Example: linear technology

3) F.O.C. conditions may not determine a unique solution. For a unique global optimum, $V(y)$ must be convex.

Comparative Statics for Conditional Factor Demand Function

By dedinition, $\mathbf{x}(\mathbf{w}, y)$ must satisfy

$$\begin{aligned} f(\mathbf{x}(\mathbf{w}, y)) &\equiv y \\ \mathbf{w} - \lambda \mathbf{D_x} f(\mathbf{x}(\mathbf{w}, y)) &\equiv 0 \end{aligned}$$

We can use these identities to find expression for $\frac{\partial x_i}{\partial w_j}$ by differentiating w.r.t. w_j.

With 2 factor inputs, diffrentiating w.r.t. w_1 gives

$$\begin{pmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial w_1} \\ \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Using Cramer's rule

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 0 & 0 & -f_2 \\ -f_1 & -1 & -\lambda f_{21} \\ -f_2 & 0 & -\lambda f_{22} \end{vmatrix}}{\begin{vmatrix} 0 & -f_1 & -f_2 \\ -f_1 & -\lambda f_{11} & -\lambda f_{21} \\ -f_2 & -\lambda f_{12} & -\lambda f_{22} \end{vmatrix}}$$

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 0 & 0 & -f_2 \\ -f_1 & -1 & -\lambda f_{21} \\ -f_2 & 0 & -\lambda f_{22} \end{vmatrix}}{\text{det of bordered Hessian}}$$

$$= \frac{f_2^2}{\text{det of bordered Hessian}} < 0 \text{ by S.O.C.}$$

In addition,

$$\frac{\partial x_2}{\partial w_1} = \frac{\begin{vmatrix} 0 & -f_1 & 0 \\ -f_1 & -\lambda f_{11} & -1 \\ -f_2 & -\lambda f_{12} & 0 \end{vmatrix}}{\text{det of bordered Hessian}}$$

$$= \frac{-f_2 f_1}{\text{det of bordered Hessian}}$$

$$= \frac{\partial x_1}{\partial w_2} = \frac{\begin{vmatrix} 0 & 0 & -f_2 \\ -f_1 & 0 & -\lambda f_{21} \\ -f_2 & -1 & -\lambda f_{22} \end{vmatrix}}{\text{det of bordered Hessian}}$$

$$= \frac{-f_1 f_2}{\text{det of bordered Hessian}}$$

$$> 0 \text{ by S.O.C. and } f_1,\ f_2 > 0$$

Hence, as a consequence of cost minimization, the cross-price effects must be equal.

For 2-input case, their signs must be positive, i.e. the two factors must be substitutes.

COST FUNCTION

Average and marginal costs

$$c(\mathbf{w}, y) \equiv \mathbf{w}\mathbf{x}(\mathbf{w}, y)$$

The minimum cost of producing y units of output is the cost of the cheapest way to produce y

Let $\mathbf{x}_f$ be the vector of fixed factors, $\mathbf{x}_v$ be the vector of variable factors

$\mathbf{w} = (\mathbf{w}_v, \mathbf{w}_f)$ be the vectors of prices of variable and fixed factors.

The short-run conditional factor demand is $\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$

Short-run Costs

short-run total cost (STC)

$$= c\left(\mathbf{w}, y, \mathbf{x}_f\right) = \mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f \mathbf{x}_f$$

short-run variable cost (SVC) $= \mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$

fixed cost (FC) $= \mathbf{w}_f \mathbf{x}_f$

short-run average cost (SAC) $= \frac{c(\mathbf{w},y,\mathbf{x}_f)}{y}$

short-run average variable cost (SAVC) $= \frac{\mathbf{w}_v \mathbf{x}_v(\mathbf{w},y,\mathbf{x}_f)}{y}$

short-run average fixed cost (SAFC) $= \frac{\mathbf{w}_f \mathbf{x}_f}{y}$

short-run marginal cost (SMC) $= \frac{\partial c(\mathbf{w},y,\mathbf{x}_f)}{\partial y}$

Long-run Costs

When all factors are variable,

let $\mathbf{x}_f(\mathbf{w}, y)$ be the optimal choice of fixed factors (now can be varied),

$\mathbf{x}_v(\mathbf{w}, y) = \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f(\mathbf{w}, y))$ be the long-run optimal choice of the variable factors, then

long-run average cost (LAC) $= \frac{c(\mathbf{w},y)}{y}$

long-run marginal cost (LMC) $= \frac{\partial c(\mathbf{w},y)}{\partial y}$

Constant returns to scale If the production function exhibits constant returns to scale, then $c\left(\mathbf{w}, y\right) = yc\left(\mathbf{w}, \mathbf{1}\right)$ and AC $=$ AVC $=$ MC

The geometry of costs

Average cost curves

In the short run, it is often thought that the average cost first decreases and then increases.

$$\begin{aligned} \text{SAC} &= \frac{c\left(\mathbf{w}, y, \mathbf{x}_f\right)}{y} = \frac{\mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y} + \frac{\mathbf{w}_f \mathbf{x}_f}{y} \\ &= \text{SAFC} + \text{SAVC} \end{aligned}$$

Assuming there are some short-run fixed factors e.g. machines, buliding ,etc., SAVC may initially fall because of economies of scale. However, as SAVC will start to rise once we approach capacity level of output. On the other hand, SAFC must decreases with output.

The level of output at which average cost is minimized is call the **minimum efficient scale**.

Marginal cost curves

Assume factor prices are fixed.

If y^* is the minimum average cost, then average costs are declining when $y \leq y^*$,i.e.

$$\frac{d}{dy}\left(\frac{c(y)}{y}\right) \leq 0 \text{ when } y \leq y^*$$

Differentiate

$$\begin{aligned} \frac{yc'(y) - c(y)}{y^2} &\leq 0 \quad \text{for } y \leq y^* \\ c'(y) &\leq \frac{c(y)}{y} \quad \text{for } y \leq y^* \end{aligned}$$

That is MC is less than AC to the left of y^* the minimum average cost.

Analogously,

$$c'(y) \geq \frac{c(y)}{y} \text{ for } y \geq y^*$$

Thus,

$$c'(y^*) \geq \frac{c(y^*)}{y^*}$$

MC = AC at minimum AC

Note also that MC of the first unit of output is equal to AC os the first unit

Long-run and short-run cost curves

Short-run cost minimization problem is a constrained version of long-run cost minimization problem. Hence, long-run cost curve lies on or below short-run cost curve.

The long-run cost is $c(y) = c(y, z(y))$
where $z(y) =$ cost-minimizing demand for a single fixed factor.
Let $z^* = z(y^*)$ be a long-run demand for the fixed factor when $y = y^*$
short-run cost $= c(y, z^*) \geq c(y, z(y)) =$ long-run cost, for all y
$c(y^*, z^*) = c(y^*, z(y^*))$ at $y = y^*$
Hence, long-run and short-run cost curves must be tangent at y^*

Alternatively, the slope of the long-run cost curve at y^* is

$$\frac{dc(y^*, z(y^*))}{dy} = \frac{\partial c(y^*, z^*)}{\partial y} + \frac{\partial c(y^*, z^*)}{\partial z}\frac{\partial z(y^*)}{\partial y}$$

but $\frac{\partial c(y^*,z^*)}{\partial z} = 0$ since z^* is the optimal at y^*, thus long-run MC $=$ short-run MC at y^*

Thus, we must also have long-run and short-run average cost curves tangent at y^*

Note also that LAC is the lower envelope of SAC curves.