COST FUNCTION

Average and marginal costs

$$c(\mathbf{w}, y) \equiv \mathbf{w}\mathbf{x}(\mathbf{w}, y)$$

The minimum cost of producing y units of output is the cost of the cheapest way to produce y

Let $\mathbf{x}_f$ be the vector of fixed factors, $\mathbf{x}_v$ be the vector of variable factors

$\mathbf{w} = (\mathbf{w}_v, \mathbf{w}_f)$ be the vectors of prices of variable and fixed factors.

The short-run conditional factor demand is $\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$

Short-run Costs

short-run total cost (STC)

$$= c\left(\mathbf{w}, y, \mathbf{x}_f\right) = \mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f\mathbf{x}_f$$

short-run variable cost (SVC) $= \mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$

fixed cost (FC) $= \mathbf{w}_f\mathbf{x}_f$

short-run average cost (SAC) $= \frac{c(\mathbf{w}, y, \mathbf{x}_f)}{y}$

short-run average variable cost (SAVC) $= \frac{\mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y}$

short-run average fixed cost (SAFC) $= \frac{\mathbf{w}_f\mathbf{x}_f}{y}$

short-run marginal cost (SMC) $= \frac{\partial c(\mathbf{w}, y, \mathbf{x}_f)}{\partial y}$

Long-run Costs

When all factors are variable,

let $\mathbf{x}_f(\mathbf{w}, y)$ be the optimal choice of fixed factors (now can be varied),

$\mathbf{x}_v(\mathbf{w}, y) = \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f(\mathbf{w}, y))$ be the long-run optimal choice of the variable factors, then

long-run average cost (LAC) $= \frac{c(\mathbf{w},y)}{y}$

long-run marginal cost (LMC) $= \frac{\partial c(\mathbf{w},y)}{\partial y}$

Constant returns to scale If the production function exhibits constant returns to scale, then $c(\mathbf{w}, y) = yc(\mathbf{w}, 1)$ and AC $=$ AVC $=$ MC

The geometry of costs

Average cost surves

In the short run, it is often thought that the average cost first decreases and then increases.

$$\begin{aligned} \text{SAC} &= \frac{c\left(\mathbf{w}, y, \mathbf{x}_f\right)}{y} = \frac{\mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y} + \frac{\mathbf{w}_f \mathbf{x}_f}{y} \\ &= \text{SAFC} + \text{SAVC} \end{aligned}$$

Assuming there are some short-run fixed factors e.g. machines, buliding ,etc., SAVC may initially because of economies of scale. However, as SAVC will start to rise once we approach capacity level of output. On the other hand, SAFC must decreases with output.

The level of output at which average cost is minimized is call the minimum efficient scale.

Marginal cost curves

Assume factor prices are fixed.

If y^* is the minimum average cost, then average costs are declining when $y \leq y^*$,i.e.

$$\frac{d}{dy}\left(\frac{c(y)}{y}\right) \leq 0 \text{ when } y \leq y^*$$

Differentiate

$$\begin{aligned} \frac{yc'(y) - c(y)}{y^2} &\leq 0 \text{ for } y \leq y^* \\ c'(y) &\leq \frac{c(y)}{y} \text{ for } y \leq y^* \end{aligned}$$

That is MC is less than AC to the left of y^* the minimum average cost.

Analogously,

$$c'(y) \geq \frac{c(y)}{y} \text{ for } y \geq y^*$$

Thus,

$$c'(y^*) \geq \frac{c(y^*)}{y^*}$$

MC $=$ AC at minimum AC

Note also that MC of the first unit of output is equal to AC os the first unit

Long-run and short-run cost curves

Short-run cost minimization problem is a constrained version of long-run cost minimization problem. Hence, long-run cost curve lies on or below short-run cost curve.

The long-run cost is $c(y) = c(y, z(y))$
where $z(y)$ = cost-minimizing demand for a single fixed factor.
Let $z^* = z(y^*)$ be a long-run demand for the fixed factor when $y = y^*$
short-run cost $= c(y, z^*) \geq c(y, z(y))$ = long-run cost, for all y
$c(y^*, z^*) = c(y^*, z(y^*))$ at $y = y^*$
Hence, long-run and short-run cost curves must be tangent at y^*

Alternatively, the slope of the long-run cost curve at y^* is

$$\frac{dc(y^*, z(y^*))}{dy} = \frac{\partial c(y^*, z^*)}{\partial y} + \frac{\partial c(y^*, z^*)}{\partial z}\frac{\partial z(y^*)}{\partial y}$$

but $\frac{\partial c(y^*,z^*)}{\partial z} = 0$ since z^* is the optimal at y^*, thus long-run MC = short-run MC at y^*

Thus, we must also have long-run and short-run average cost curves tangent at y^*

Note also that LAC is the lower envelope of SAC curves.

Factor prices and cost functions

Properties of the cost function

1) Nondecreasing in $\mathbf{w}$. If $\mathbf{w}' \geq \mathbf{w}$, then $c(\mathbf{w}', y) \geq c(\mathbf{w}, y)$

Proof. Let $\mathbf{x}$ and $\mathbf{x}'$ be cost-minimizing bundles associated with $\mathbf{w}$ and $\mathbf{w}'$. Then, $\mathbf{wx} \leq \mathbf{wx}'$ by cost minimization and $\mathbf{wx}' \leq \mathbf{w}'\mathbf{x}'$as $\mathbf{w} \leq \mathbf{w}'$. Hence, $\mathbf{wx} \leq \mathbf{w}'\mathbf{x}'$

2) Homogeneous of degree 1 in $\mathbf{w}$. $c(t\mathbf{w}, y) = tc(\mathbf{w}, y)$ for $t > 0$

Proof. If $\mathbf{x}$ is a cost-minization bundle at price $\mathbf{w}$, then $\mathbf{x}$ must also minimize cost at prices $t\mathbf{w}$. Suppose not and let $\mathbf{x}'$ minimize cost at $t\mathbf{w}$ then $t\mathbf{wx}' < t\mathbf{wx}$ but this means $\mathbf{wx}' < \mathbf{wx}$ cintradicting our assumption that $\mathbf{x}$ is a cost-minization bundle at price $\mathbf{w}$.

3) Concave in $\mathbf{w}$. $c(t\mathbf{w}+(1-t)\mathbf{w}', y) \geq tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}', y)$ for $0 \leq t \leq 1$

Proof. Let $(\mathbf{w}, \mathbf{x})$, $(\mathbf{w}', \mathbf{x}')$ and $(\mathbf{w}'', \mathbf{x}'')$be combinations of prices and input bundles that minimizes costs where $\mathbf{w}'' \equiv t\mathbf{w} + (1-t)\mathbf{w}'$ then $c(\mathbf{w}'', y) = \mathbf{w}''\mathbf{x}'' = t\mathbf{w}\mathbf{x}'' + (1-t)\mathbf{w}'\mathbf{x}''$. Note that $\mathbf{w}\mathbf{x}'' \geq c(\mathbf{w}, y)$ and $\mathbf{w}'\mathbf{x}'' \geq c(\mathbf{w}', y)$. Thus $c(\mathbf{w}'', y) = c(t\mathbf{w} + (1-t)\mathbf{w}', y) \geq tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}', y)$

4) The cost function is continuous (see Theorem of the Maximum)

Shephard's lemma: Let $x_i(\mathbf{w}, y)$ be the firm's conditional factor demand for input i. Then if the cost function is differentiable at $(\mathbf{w}, y)$, and $w_i > 0$ for $i = 1, ...n$ then

$$x_i(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_i} \qquad i = 1, ..., n$$

Proof. Let $\mathbf{x}^*$ be a cost-minimizing bundle that produces y at prices $\mathbf{w}^*$ and define

$g(\mathbf{w}) = c(\mathbf{w}, y) - \mathbf{w}\mathbf{x}^* \leq 0$ as $c(\mathbf{w}, y)$ is the cheapest way to produce y.

Note that $g(\mathbf{w}^*) = 0$ is the maximum value, hence

$$\frac{\partial g(\mathbf{w}^*)}{\partial w_i} = \frac{\partial c(\mathbf{w}^*, y)}{\partial w_i} - x_i^* = 0 \qquad i = 1, ..., n$$

The envelope theorem for constrained optimization

A general problem with n inputs

Define

$$\begin{aligned} M(a) &\equiv \max_{\mathbf{x}} g(\mathbf{x}, a) \\ \text{s.t. } h(\mathbf{x}, a) &= 0 \end{aligned}$$

The Lagrangian is

$$L = g(\mathbf{x}, a) - \lambda h(\mathbf{x}, a)$$

and associated F.O.C.s are

$$\begin{aligned} \frac{\partial g}{\partial x_i} - \lambda \frac{\partial h}{\partial x_i} &= 0 \qquad i = 1, ..., n \\ h(\mathbf{x}, a) &= 0 \end{aligned}$$

Denote the solution of this problem by $\mathbf{x}(a)$.Remark that $\mathbf{x}(a)$ must satisfy the above F.O.C.s and we have

$$M(a) = g(\mathbf{x}(a), a)$$

Differentiate w.r.t. a

$$\frac{dM(a)}{da} = \frac{\partial g}{\partial x_1}\frac{dx_1}{da} + ... + \frac{\partial g}{\partial x_n}\frac{dx_n}{da} + \frac{\partial g}{\partial a}$$

From F.O.C.s

$$\frac{dM(a)}{da} = \lambda\left[\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + ... + \frac{\partial h}{\partial x_n}\frac{dx_n}{da}\right] + \frac{\partial g}{\partial a}$$

Note also that

$$h(\mathbf{x}(a), a) \equiv 0$$

Differentiate w.r.t. a

$$\left[\frac{\partial h}{\partial x_1}\frac{dx_1}{da} + ... + \frac{\partial h}{\partial x_n}\frac{dx_n}{da}\right] + \frac{\partial h}{\partial a} = 0$$

Hence,

$$\begin{aligned}\frac{dM(a)}{da} &= \frac{\partial g(\mathbf{x}, a)}{\partial a}|_{\mathbf{x}=\mathbf{x}(a)} - \lambda\frac{\partial h(\mathbf{x}, a)}{\partial a}|_{\mathbf{x}=\mathbf{x}(a)} \\ &= \frac{\partial L(\mathbf{x}, a)}{\partial a}|_{\mathbf{x}=\mathbf{x}(a)}\end{aligned}$$

Apply this result to our cost minimization problem,

$M(a) = c(\mathbf{w}, y),$

$g(\mathbf{x}, a) = w_1 x_1 + ... + w_n x_n,$

and $h(\mathbf{x}, a) = f(\mathbf{x}) - y$

$$
\begin{aligned}
\frac{\partial c(\mathbf{w}, y)}{\partial w_i} &= \frac{\partial L(\mathbf{x}, \mathbf{w}, y)}{\partial w_i}|_{x_i = x_i(\mathbf{w}, y)} \\
&= x_i|_{x_i = x_i(\mathbf{w}, y)} = x_i(\mathbf{w}, y)
\end{aligned}
$$

This is the Shepard's lemma.

Comparative statics

Because of Shepard's lemma, certain properties of the cost function translate into conditional factor demand functions

1) Since the cost function is nondecreasing in factor prices, $\frac{\partial c(\mathbf{w},y)}{\partial w_i} = x_i(\mathbf{w},y) > 0$

2) The cost function is homogeneous of degree 1 in $\mathbf{w}$. Therefore the derivatives of the cost function, the factor demands are homogeneous of degree 0 in $\mathbf{w}$.

3) The cost function is concave in w. In the case of two inputs

$$\begin{pmatrix} \frac{\partial c^2(\mathbf{w},y)}{\partial w_i^2} & \frac{\partial c^2(\mathbf{w},y)}{\partial w_i \partial w_j} \\ \frac{\partial c^2(\mathbf{w},y)}{\partial w_j \partial w_i} & \frac{\partial c^2(\mathbf{w},y)}{\partial w_j^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_i(\mathbf{w},y)}{\partial w_i} & \frac{\partial x_i(\mathbf{w},y)}{\partial w_j} \\ \frac{\partial x_j(\mathbf{w},y)}{\partial w_i} & \frac{\partial x_j(\mathbf{w},y)}{\partial w_j} \end{pmatrix}$$

is a symmetric negative semidefinite matrix. Thus,

a) the cross-price effects are symmetric

$$\frac{\partial x_i(\mathbf{w},y)}{\partial w_j} = \frac{\partial c^2(\mathbf{w},y)}{\partial w_i \partial w_j} = \frac{\partial c^2(\mathbf{w},y)}{\partial w_j \partial w_i} = \frac{\partial x_j(\mathbf{w},y)}{\partial w_i}$$

b) the own-price effects are nonpositive.

$$\frac{\partial x_i(\mathbf{w},y)}{\partial w_i} = \frac{\partial c^2(\mathbf{w},y)}{\partial w_i^2} \leq 0$$

as the diagonal terms of a negative semidefinite must be nonpositive.The conditional demand curves are downward slopng.

c) The vector of changes in factor demands moves 'opposite' the vector of changes in factor prices

$$\mathbf{dwdx} \leq 0$$

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