Duality

Let $V^*(y) = \{\mathbf{x} : \mathbf{wx} \geq \mathbf{wx(w,}y) = c(\mathbf{w,}y)$ for all $\mathbf{w} \geq \mathbf{0}\}$

and $c^*(\mathbf{w}, y) = \min \mathbf{wx}$ such that $\mathbf{x}$ is in $V^*(y)$

Then, given a cost function we can construct an input requirement set $V^*(y)$.

i) If the original technology is convex and monotonic, the constructed $V^*(y)$ will be equal to the original technology $V(y)$ and $c^*(\mathbf{w}, y) = c(\mathbf{w}, y)$ the cost function of the original technology.

ii) If the original technology is nonconvex and nonmonotonic, $V^*(y) \neq V(y)$ but we still have $c^*(\mathbf{w}, y) = c(\mathbf{w}, y)$.

Sufficient conditons for cost funcitons

Given a function that has the properties of the cost function, i.e. nondecreasing, homogeneous, concave and continuous function of prices, it is necessarily the cost function of some technology.

Let $\phi(\mathbf{w}, y)$ be a differentiable function satisfying

1) homogeneous of degree 1, $\phi(t\mathbf{w}, y) = t\phi(\mathbf{w}, y)$ for all $t \geq 0$

2) $\phi(\mathbf{w}, y) \geq 0$ for $\mathbf{w} \geq 0$ and $y \geq 0$

3) Monotonic, $\phi(\mathbf{w}', y) \geq \phi(\mathbf{w}, y)$ for $\mathbf{w}' \geq \mathbf{w}$

4) Concave in $\mathbf{w}$, $\phi(t\mathbf{w} + (1-t)\mathbf{w}', y) \geq t\phi(\mathbf{w}, y) + (1-t)\phi(\mathbf{w}', y)$ for $0 \leq t \leq 1$

Then $\phi(\mathbf{w}, y)$ is a cost function for the technology defined by $V^*(y) = \{\mathbf{x} \geq \mathbf{0} : \mathbf{wx} \geq \phi(\mathbf{w}, y) \text{ for all } \mathbf{w} \geq \mathbf{0}\}$

Demand fucntions

Given a set of functions that satisfy the properties of the conditional factor demand functions, then the set of functions are necessarily conditional factor demand functions of some technology. In particular assume

i) a set of functions $(g_i(\mathbf{w}, y))$ are homogeneous of degree 0 in prices and

ii) $\left(\frac{\partial g_i(\mathbf{w},y)}{\partial w_j}\right)$ is a symmetric negative semidefintie matrix

Then, we can derive a function $\phi(\mathbf{w}, y) = \sum_{i=1}^{n} w_i g_i(\mathbf{w}, y)$ with the properties of the cost function and hence there is a technology $V^*(y)$.that yields $(g_i(\mathbf{w}, y))$ as its conditional factor demand functions.

Geometry of duality

Isocost curve gives all the combinations of prices that minimises cost at a particular level given a specific level of output is produced.

Isoquant curve gives all input bundles that produces a particular level of output.

The slope of an isocost curve at point (w_1^*, w_2^*) is

$$\frac{dw_2(w_1^*)}{dw_1} = -\frac{\frac{\partial c(\mathbf{w}^*, y)}{\partial w_1}}{\frac{\partial c(\mathbf{w}^*, y)}{\partial w_2}} = -\frac{x_1(\mathbf{w}^*, y)}{x_2(\mathbf{w}^*, y)}$$

The second equality is due to derivative property.

The slope of an isoquant at a point (x_1^*, x_2^*) is

$$\frac{dx_2(x_1^*)}{dx_1} = -\frac{\frac{\partial f(\mathbf{x}^*)}{\partial x_1}}{\frac{\partial f(\mathbf{x}^*)}{\partial x_2}} = -\frac{w_1^*}{w_2^*}$$

The second equality is due to the first order conditions of cost minimization