UTILITY MAXIMIZATION

Consumer Preferences

The following properties allow preferences to order the set of bundles. For any consumption bundles $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in consumption set X, preferences are

COMPLETE: For all $\mathbf{x}$ and $\mathbf{y}$ in X, either $\mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$ or both

REFLEXIVE: For all $\mathbf{x}$ in X, $\mathbf{x} \succeq \mathbf{x}$.

TRANSITIVE: For all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in X, if $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{y} \succeq \mathbf{z}$, then $\mathbf{x} \succeq \mathbf{z}$

The above three properties and continuity ensure that consumer's behaviour can be represented by a continuous utility function.

CONTINUITY: For all $\mathbf{y}$ in X, the sets $\{\mathbf{x} : \mathbf{x} \succeq \mathbf{y}\}$ or $\{\mathbf{x} : \mathbf{x} \preceq \mathbf{y}\}$ are closed sets. It follows that or $\{\mathbf{x} : \mathbf{x} \succ \mathbf{y}\}$ or $\{\mathbf{x} : \mathbf{x} \prec \mathbf{y}\}$ are open sets.

Utility function has an ordinal property. If $u(\mathbf{x})$ represents some preferences $\succeq$ and $f : R \to R$ is a monotonic function, then $f(u(\mathbf{x}))$ will represent exactly the same preferences since $f(u(\mathbf{x})) \geq f(u(\mathbf{y}))$ iff $u(\mathbf{x}) \geq u(\mathbf{y})$.

Other useful assumptions

LOCAL NONSATIATION: Given any $\mathbf{x}$ in X and any $\epsilon > 0$, then there is some bundle $\mathbf{y}$ in X with $|\mathbf{x} - \mathbf{y}| < \epsilon$ such that $\mathbf{y} \succ \mathbf{x}$. (One can do a little bit better, rules out 'thick' indifference curve)

CONVEXITY: Given $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in X such that $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, then it follows that $t\mathbf{x} + (1 - t)\mathbf{y} \succeq \mathbf{z}$ for all $0 \leq t \leq 1$. (One prefers average to extreme, the indifference curve may have flat parts)

STRICT CONVEXITY: Given $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{z}$ in X such that $\mathbf{x} \succeq \mathbf{z}$ and $\mathbf{y} \succeq \mathbf{z}$, then $t\mathbf{x} + (1 - t)\mathbf{y} \succ \mathbf{z}$ for all $0 \leq t \leq 1$. (Indifference curve are strictly rotund)

The Marginal Rate of Substitution

The marginal rate of substitution shows how does a consumption of one good has to change in response to an increase in consumption of another good holding utility constant.

Since utility is held constant, we have

$$\frac{\partial u(\mathbf{x})}{\partial x_i} dx_i + \frac{\partial u(\mathbf{x})}{\partial x_j} dx_j = 0$$

$$\text{MRS betwen goods } i \text{ and } j = \frac{dx_j}{dx_i} = -\frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{\frac{\partial u(\mathbf{x})}{\partial x_j}}$$

Note that MRS does not depend on the utility function chosen to represent the underlying preferences. If $g(u)$ is a monotonic transformation of utility.

$$\frac{dx_j}{dx_i} = -\frac{g'(u)\frac{\partial u(\mathbf{x})}{\partial x_i}}{g'(u)\frac{\partial u(\mathbf{x})}{\partial x_j}} = -\frac{\frac{\partial u(\mathbf{x})}{\partial x_i}}{\frac{\partial u(\mathbf{x})}{\partial x_j}}$$

Consumer behaviour

Let m be the fixed amount of money available to a consumer, and let $\mathbf{p} = (p_1, ..., p_k)$ be the vector of prices of goods $1, ..., k$. The preference maximization problem is

$$\max u(\mathbf{x})$$

such that $\mathbf{px} \leq m$

Under local nonsatiation, if $\mathbf{x}^*$ is the maximizing choice then we must have $\mathbf{px}^* = m$. Suppose not, then $\mathbf{px}^* < m$, and bundles close enough to $\mathbf{x}^*$ also cost less than m. Under local nonsatiation, there must then be some feasible bundle $\mathbf{x}$ (close to $\mathbf{x}^*$) which is prefered to $\mathbf{x}^*$. Hence, we can rewrite the problem as

$$\max u(\mathbf{x})$$

such that $\mathbf{px} = m$

Note

i) the maximizing choice $\mathbf{x}^*$ will be independent of the choice of the utility function used to represent the preferences. Different utility functions representing exactly the same preference will pick out $\mathbf{x}^*$ as the optimal choice.

ii) the optimal choice set is 'homogeneous of degree zero' in $(\mathbf{p}, m)$. If $\mathbf{x}^* \succeq \mathbf{x}$ for all $\mathbf{x}$ such that $\mathbf{px} \leq m$, then $\mathbf{x}^* \succeq \mathbf{y}$ for all $\mathbf{y}$ such that $t\mathbf{py} \leq tm$.

iii) strict convexity of preferences will ensure a unique solution.

Solution

The Lagrangian $L = u(\mathbf{x}) - \lambda(\mathbf{px} - m)$

F.O.C.: $\frac{\partial u(\mathbf{x})}{\partial x_i} - \lambda p_i = 0$ for $i = 1, .., k$
Divide i^{th} condition by j^{th} condition, we get

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}} = \frac{p_i}{p_j} \text{ for } i,j = 1,..,k$$

This says that MRS must be equal to the price ratio which represent the **economic rate of substitution**

S.O.C.: For strict maximum, $\mathbf{h}^T D^2_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*)\mathbf{h} < 0$ subject to the constraint $D\mathbf{g}(\mathbf{x}^*)\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$ where $\mathbf{g}(\mathbf{x}^*) = (\mathbf{p}\mathbf{x}^* - m)$.

Note that $D^2_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = D^2 u(\mathbf{x}^*)$ and $D\mathbf{g}(\mathbf{x}^*) = \mathbf{p}$.

Hence, we must have $\mathbf{h}^T D^2 u(\mathbf{x}^*)\mathbf{h} < 0$ subject to the constraint $\mathbf{p}\mathbf{h} = 0$ for all $\mathbf{h} \neq \mathbf{0}$. We can look at conditions involving bordered Hessian to check for S.O.C.

For negative definiteness: the bordered-preserving principal minor of order k has the sign $(-1)^k$ for $k = 2, 3, ..., n$

Indirect Utility Function and Consumer's Demand Function

Define **Indirect Utility Function** as

$$v(\mathbf{p}, m) = \max u(\mathbf{x})$$

such that $\mathbf{px} = m$

$v(\mathbf{p}, m)$ gives the maximum utility achievable at given prices and income.

The value $\mathbf{x}$ that solve this problem is called consumer's demanded bundle.

Consumer's demand function is the function $\mathbf{x}(\mathbf{p}, m)$ that gives the optimal bundle as a function of $\mathbf{p}$ and m. The consumer's demand function is homogeneous of degree zero' in $(\mathbf{p}, m)$.

Properties of the Indirect Utility Function

1) $v(\mathbf{p}, m)$ is nonincreasing in $\mathbf{p}$; if $\mathbf{p}' \geq \mathbf{p}, v(\mathbf{p}', m) \leq v(\mathbf{p}, m)$. and
$v(\mathbf{p}, m)$ is nondecreasing in m.

2) $v(\mathbf{p}, m)$ is homogeneous of degree zero in $(\mathbf{p}, m)$.

3) $v(\mathbf{p}, m)$ is quasiconvex in $\mathbf{p}$; that is $\{\mathbf{p} : v(\mathbf{p}, m) \leq k\}$ (lower contour set) is a convex set for all k.

4) $v(\mathbf{p}, m)$ is continuous at all $\mathbf{p} \gg \mathbf{0}$,$m > 0$.

Note that if preferences satisfy the local nonsatiation assumption, then $v(\mathbf{p}, m)$ will be strictly increasing in m.

The Expenditure Function

The expenditure function $e(\mathbf{p}, u)$.is the inverse of $v(m) |_{\mathbf{p}}$ It gives the minimum amount of income necessary to achieve utility u at prices $\mathbf{p}$.

Alternatively,

$$\begin{aligned} e(\mathbf{p}, u) &= \min \mathbf{px} \\ \text{s.t. } u(\mathbf{x}) &\geq u \end{aligned}$$

It gives the minimum cost of achieving a fixed level of utility u at prices $\mathbf{p}$.

Properties of the expenditure function (analogous to the cost function)

1) $e(\mathbf{p}, u)$ is nondecreasing in $\mathbf{p}$.
2) $e(\mathbf{p}, u)$ is homogeneous of degree 1in $\mathbf{p}$.
3) $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$.
4) $e(\mathbf{p}, u)$ is continuous in $\mathbf{p}$.

5) If $h(\mathbf{p}, u)$ is the expenditure-minimizing bundle necessary to achieve utility level u at prices $\mathbf{p}$, then

$$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i} \text{ for } i = 1, ..., k$$

assuming derivative exists and $p_i > 0$.

Hicksian demand function

(analogous to conditional factor demand function)

The expenditure-minimizing bundle necessary to achieve utility level u at prices $\mathbf{p}$ denoted by $h(\mathbf{p}, u)$ is called the **Hicksian Demand function** or **Compensated Demand functuion**.

It tells us what consumption bundle achieve a target level of utility and minimizes total expenditure.

Marshallian demand function

This is the ordinary consumer's demand function $\mathbf{x}(\mathbf{p}, m)$.

Important Identities

First consider the utility maximization problem

$$v(\mathbf{p}, m^*) = \max u(\mathbf{x})$$

such that $\mathbf{px} \leq m^*$
Let $\mathbf{x}^*$ be the solution to this problem and let $u^* = u(\mathbf{x}^*)$

Then consider the expenditure minimization problem

$$e(\mathbf{p}, u^*) = \min \mathbf{px}$$

such that $u(\mathbf{x}) \geq u^*$
The solution to this problem should also be $\mathbf{x}^*$.

We have,
1) $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m$
2) $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u$
3) $x_i(\mathbf{p}, m) \equiv h_i(\mathbf{p}, v(\mathbf{p}, m))$
4) $h_i(\mathbf{p}, u) \equiv x_i(\mathbf{p}, e(\mathbf{p}, u))$

4) says that the Hicksian demand function is equal to the Marshallian demand function if the consumer's income is compensated so as to achieve some target level of utility.

Roy's Identity : If $\mathbf{x}(\mathbf{p}, m)$ is the Marshallian demand function, then

$$x_i(\mathbf{p}, m) = \frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_i}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}} \text{ for } i = 1, .., k$$

provided that the RHS is well defined and $p_i > 0$ and $m > 0$.

Proof. If $\mathbf{x}^*$ yields maximum utility of u^* at $\mathbf{p}^*$ and m^*
We know that $\mathbf{x}(\mathbf{p}^*, m^*) \equiv \mathbf{h}(\mathbf{p}^*, u^*)$
and $v(\mathbf{p}, e(\mathbf{p}, u^*)) \equiv u^*$

Differentiate the latter wrt p_i to get
$0 = \frac{\partial v(\mathbf{p}^*, m^*)}{\partial p_i} + \frac{\partial v(\mathbf{p}^*, m^*)}{\partial m} \frac{\partial e(\mathbf{p}, u^*)}{\partial p_i}$
Hence,

$\mathbf{x}(\mathbf{p}^*, m^*) \equiv \mathbf{h}(\mathbf{p}^*, u^*) \equiv \frac{\partial e(\mathbf{p}, u^*)}{\partial p_i} \equiv -\frac{\partial v(\mathbf{p}^*, m^*)/\partial p_i}{\partial v(\mathbf{p}^*, m^*)/\partial m}$
Note that the above is satisfied for all $\mathbf{p}^*$ and m^* and $\mathbf{x}^* = \mathbf{x}(\mathbf{p}^*, m^*)$.

The Money Metric Utility Functions

(Also known as the Minimum Income Function)

Money Metric Utility gives the minimum expenditure at prices $\mathbf{p}$ necessary to purchase a bundle at least as good as $\mathbf{x}$ To answer this, we solve

$$\begin{aligned} m(\mathbf{p}, \mathbf{x}) &\equiv \min_{\mathbf{z}} \mathbf{pz} \\ \text{s.t. } u(\mathbf{z}) &\geq u(\mathbf{x}) \end{aligned}$$

or

$$m(\mathbf{p}, \mathbf{x}) \equiv e(\mathbf{p}, u(\mathbf{x}))$$

Note

i) For fixed $\mathbf{x}$, $u(\mathbf{x})$ is fixed, so $m(\mathbf{p}, \mathbf{x})$ behaves exactly like an expenditure function (hence it is monotonic, homogeneous and concave in $\mathbf{p}$)

ii) For fixed $\mathbf{p}$, $m(\mathbf{p}, \mathbf{x})$ is a monotonic transformaton of $u(\mathbf{x})$ and hence it is also a utility function

The Money Metric Indirect Utility Functions

The Money Metric Indirect Utility gives the minimum expenditure at prices $\mathbf{p}$ for consumer to be as well off as he would be facing prices $\mathbf{q}$ with income m.

$$\mu(\mathbf{p}; \mathbf{q}, m) \equiv e(\mathbf{p}, v(\mathbf{q}, m))$$

Similar to the money metric utility function,

i) For given $(\mathbf{q}, m)$, $\mu(\mathbf{p}; \mathbf{q}, m)$ behaves like an expenditure function.

ii) For fixed $\mathbf{p}$, $\mu(\mathbf{p}; \mathbf{q}, m)$ behaves like an indirect utility function with respect to $(\mathbf{q}, m)$. It is a monotonic transformation of $v(\mathbf{q}, m)$.