CHOICE

INCOME EXPANSION PATH: The locus of utility-maximizing bundles when income varies holding prices fixed.

ENGEL CURVE: The function that relates income to the demand of each good.

- If the income expansion path is a straight line through the origin, the consumer has **a unit income elasticity**. He consumes the same proportion of each commodity at each income level.

- If the income expansion path bends towards one good, that good is a **luxury good** while the other is a **necessary good**. As income increases, the consumers consumes more of both goods (hence both goods are **normal goods**) but proportionally more of the luxury good than of the necessary good.

- If the income expansion path bends bacwards and away from a good, that good is an **inferior good**. The consumers consumes less of that good as income increases.

PRICE OFFER CURVE: The locus of utility-maximizing bundles when the price of one good varies holding other prices and income fixed.

- If demand for a good increases as its price decreases, the good is an ordinary good.

- If the offer curve bends backwards, a decrease in the price of a good leads to a decrease in demand of that good.

The Slutsky equation

Slutsky equation:

$$\frac{\partial x_j(\mathbf{p},m)}{\partial p_i} = \frac{\partial h_j(\mathbf{p},v(\mathbf{p},m))}{\partial p_i} - \frac{\partial x_j(\mathbf{p},m)}{\partial m} x_i(\mathbf{p}, m)$$

Proof. If $\mathbf{x}^*$ yields maximum utility of $u^* = u(\mathbf{x}^*)$ at $\mathbf{p}^*$ and m^*

We know that $\mathbf{h}(\mathbf{p}, u^*) \equiv \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u^*))$

Differentiate wrt p_i and evaluate at $\mathbf{p}^*$ to get

$$\frac{\partial h_j(\mathbf{p}^*,u^*)}{\partial p_i} = \frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial m}\frac{\partial e(\mathbf{p},u^*)}{\partial p_i}$$

or $\frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial p_i} = \frac{\partial h_j(\mathbf{p}^*,u^*)}{\partial p_i} - \frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial m} x_i^*$

The slutsky equation decomposes the demand change Δx_j as a result of the price change Δp_i into income and substitution effect

$$\Delta x_j \approx \frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial p_i}\Delta p_i = \frac{\partial h_j(\mathbf{p}^*,u^*)}{\partial p_i}\Delta p_i - \frac{\partial x_j(\mathbf{p}^*,m^*)}{\partial m} x_i^* \Delta p_i$$

Properties of Demand Functions

1) The matrix $\left(\frac{\partial h_j(\mathbf{p},u)}{\partial p_i}\right)$ is a negative semidefinite matrix.

First, recall that $e(\mathbf{p},u)$ is concave in $\mathbf{p}$ and $h_j(\mathbf{p},u) = \frac{\partial e(\mathbf{p},u)}{\partial p_j}$,

hence $\left(\frac{\partial h_j(\mathbf{p},u)}{\partial p_i}\right) = \left(\frac{\partial^2 e(\mathbf{p},u)}{\partial p_j \partial p_i}\right)$ which is a negative semidefintie matrix.

2) The matrix of substitution term is symmetric since

$$\left(\frac{\partial h_j(\mathbf{p},u)}{\partial p_i}\right) = \left(\frac{\partial^2 e(\mathbf{p},u)}{\partial p_j \partial p_i}\right) = \left(\frac{\partial^2 e(\mathbf{p},u)}{\partial p_i \partial p_j}\right) = \left(\frac{\partial h_i(\mathbf{p},u)}{\partial p_j}\right).$$

3) The hicksian demand curve slopes downward as the compensated own-price effect is nonpositive

$$\left(\frac{\partial h_i(\mathbf{p},u)}{\partial p_i}\right) = \left(\frac{\partial^2 e(\mathbf{p},u)}{\partial p_i^2}\right) \leq 0$$

since a negative semidefinite matrix has nonpositive diagonal terms.

4) The substitution matrix $\left(\frac{\partial x_j(\mathbf{p},m)}{\partial p_i} + \frac{\partial x_j(\mathbf{p},m)}{\partial m} x_i\right)$ is a symmetric semidefintie matrix.

Comparative Statics using F.O.C.s

Consider the case with two goods, F.O.C.s are

$p_1x_1(p_1,p_2,m)+p_2x_2(p_1,p_2,m)-m\equiv 0$

$\frac{\partial u(x_1(p_1,p_2,m),x_2(p_1,p_2,m))}{\partial x_1}-\lambda p_1\equiv 0$

$\frac{\partial u(x_1(p_1,p_2,m),x_2(p_1,p_2,m))}{\partial x_2}-\lambda p_2\equiv 0$

Differentiate w.r.t. p_1 and rearrange in matrix form, we have

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial p_1} \\ \frac{\partial x_1}{\partial p_1} \\ \frac{\partial x_2}{\partial p_1} \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ \lambda \\ 0 \end{pmatrix}$$

Solving for $\frac{\partial x_1}{\partial p_1}$ via the Cramer's rule

$$\frac{\partial x_1}{\partial p_1}=\frac{\begin{vmatrix} 0 & x_1 & -p_2 \\ -p_1 & \lambda & u_{12} \\ -p_2 & 0 & u_{22} \end{vmatrix}}{H}$$

where H is determinant of bordered Hessian and is greater than zero by S.O.C.

Expanding this by cofactors on the second column, we have

$$\frac{\partial x_1}{\partial p_1} = \lambda \frac{\begin{vmatrix} 0 & -p_2 \\ -p_2 & u_{22} \end{vmatrix}}{H} - x_1 \frac{\begin{vmatrix} -p_1 & u_{12} \\ -p_2 & u_{22} \end{vmatrix}}{H}$$

Now, differentiate F.O.C.s again but this time w.r.t. m

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial m} \\ \frac{\partial x_1}{\partial m} \\ \frac{\partial x_2}{\partial m} \end{pmatrix} \equiv \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

By Cramer's rule

$$\frac{\partial x_1}{\partial m} = \frac{\begin{vmatrix} -p_1 & u_{12} \\ -p_2 & u_{22} \end{vmatrix}}{H}$$

In addition, by setting up expenditure minimization problem, we can show that $\frac{\partial h_1}{\partial p_1} = \lambda \frac{\begin{vmatrix} 0 & -p_2 \\ -p_2 & u_{22} \end{vmatrix}}{H}$