Introduction to Unconstraint Optimization

Maximise or Minimize $z(x_1, x_2, ...x_n)$

First-order conditions

If $(\mathbf{x}^*)$ is the solution to the above problem, we must have

$$
\begin{aligned}
\frac{\partial z(\mathbf{x}^*)}{\partial x_1} &= 0 \\
&. \\
&. \\
\frac{\partial z(\mathbf{x}^*)}{\partial x_n} &= 0
\end{aligned}
$$

Second-order conditions

1) For maximization problem:

Formally, the Hessian of z w.r.t. $\mathbf{x}$ at $(\mathbf{x}^*)$, denoted by $D^2z(\mathbf{x}^*)$,

is negative semidefinite, i.e. $\mathbf{h}^T D^2 z(\mathbf{x}^*)\mathbf{h} \leq 0$ for all $\mathbf{h} \neq \mathbf{0}$.

or negative definite, i.e. $\mathbf{h}^T D^2 z(\mathbf{x}^*)\mathbf{h} < 0$ for all $\mathbf{h} \neq \mathbf{0}$. (for a strict local max)

2) For minimization problem:

Formally, the Hessian of z w.r.t. $\mathbf{x}$ at $(\mathbf{x}^*)$, denoted by $D^2z(\mathbf{x}^*)$,

is positive semidefinite,i.e. $\mathbf{h}^T D^2 z(\mathbf{x}^*)\mathbf{h} \geq 0$ for all $\mathbf{h} \neq \mathbf{0}$

or positive definite,i.e. $\mathbf{h}^T D^2 z(\mathbf{x}^*)\mathbf{h} > 0$ for all $\mathbf{h} \neq \mathbf{0}$ (for a strict local min)

Checking for SOC.

1) For maximization problem:

For negative semidefinite: check that all nonzero principal minors of the Hessian matrix of order k has the sign $(-1)^k$ for $k = 1, 2, ..., n$

For negative definite: check that the leading principal minors of the Hessian matrix of order k has the sign $(-1)^k$ for $k = 1, 2, ..., n$

2) For minimization problem:

For positive semidefinite: check that all the principal minors of the Hessian matrix are nonnegative

For positive definite: check that the leading principal minors of the Hessian matrix are all positive.

Profit Maximization

For Profit Maximization with 2 inputs,

$$\max \pi = pf\left(x_1, x_2\right) - w_1 x_1 - w_2 x_2$$

F.O.C.s are

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_1} - w_1 = 0$$

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_2} - w_2 = 0$$

S.O.C. is

$$D^2\pi(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial^2 \pi(\mathbf{x}^*)}{\partial x_1^2} & \frac{\partial^2 \pi(\mathbf{x}^*)}{\partial x_2 \partial x_1} \\ \frac{\partial^2 \pi(\mathbf{x}^*)}{\partial x_1 \partial x_2} & \frac{\partial^2 \pi(\mathbf{x}^*)}{\partial x_2^2} \end{pmatrix} = p \begin{pmatrix} \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{x}^*)}{\partial x_2^2} \end{pmatrix}$$

is negative definite