Economic Risk and Decision Analysis for Oil and Gas Industry

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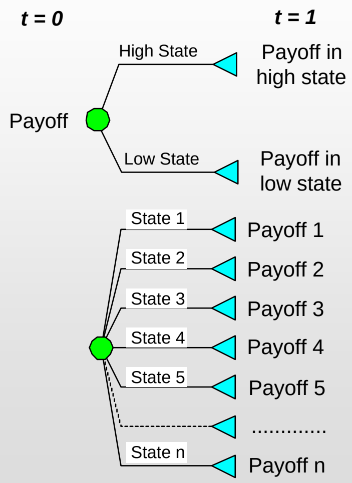
Stochastic Processes

Modeling Simple Options

- There are four main types of stochastic processes:
  - Arithmetic Brownian Motion
  - Geometric Brownian Motion
  - Mean Reverting Processes
  - Jump Processes

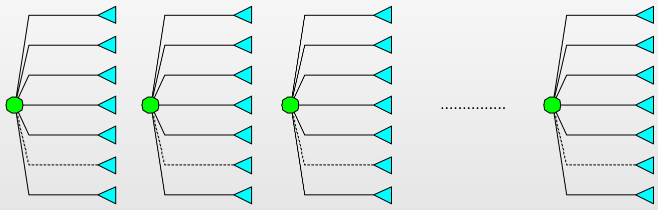
Modeling Uncertainty

- Up to now we have modeled uncertainty as having only **two possible states of nature**.
- This is actually only a **simplification of reality**, as a uncertain variable may have an infinite number of states.
- A more accurate model as the one to the right shows the corresponding payoffs for each possible state that may occur one period from now.



Modeling Uncertainty

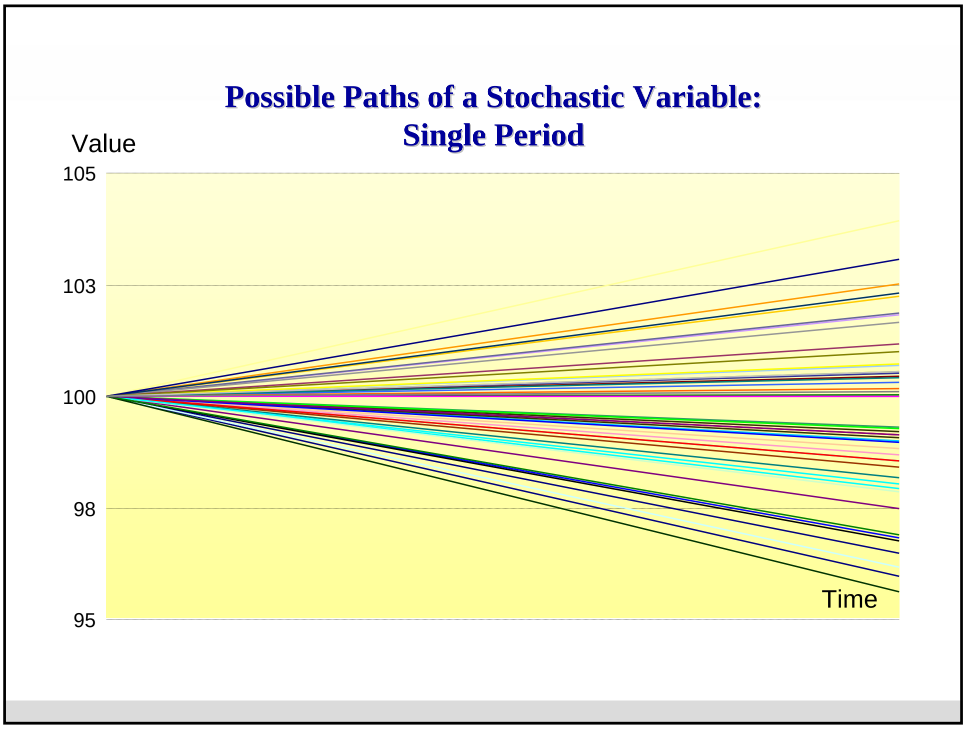
- For most problems, we will also need to expand this to include **multiperiod models of the relevant variables**.

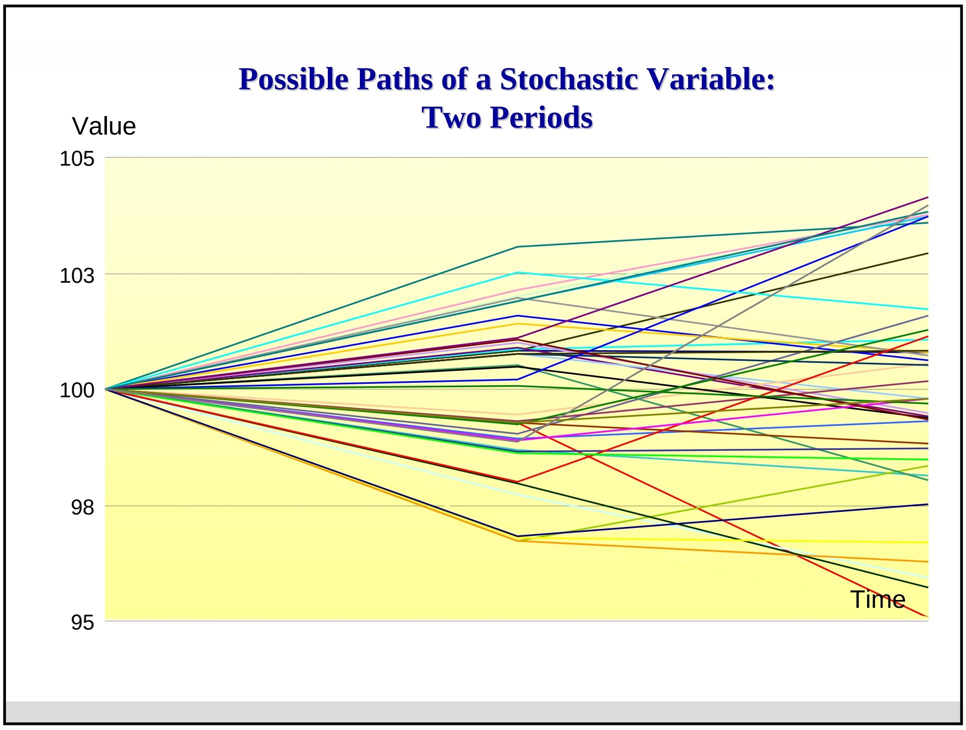


- One way to obtain a more detailed model of the uncertainty is to assume that the variable of interest follows a **stochastic**, or **random process**.

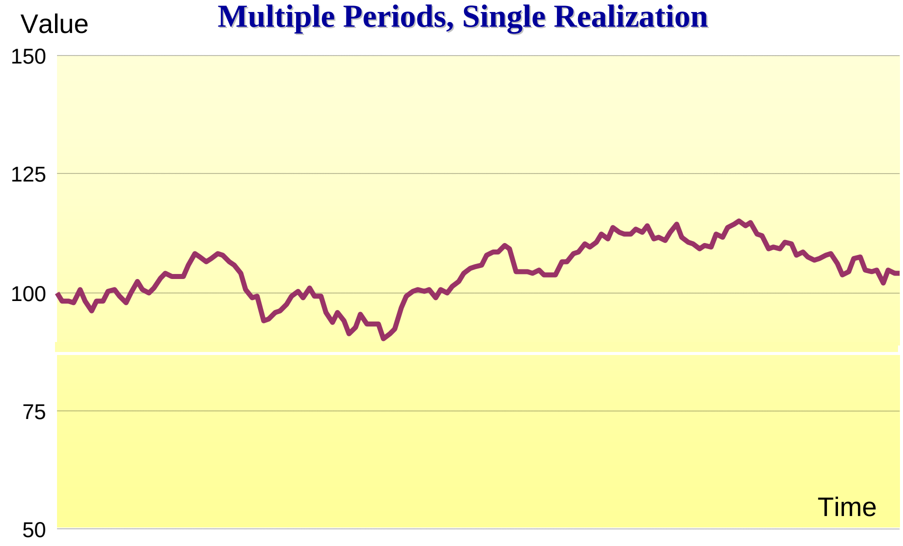
Stochastic Processes

- A **Stochastic Process** is a variable that evolves over time in a way that is ***at least partially random***. Ex:
  - Price of stock
  - Monthly consumption of energy in the US.
- A Stochastic Process may be:
  - **Stationary** – when the statistical properties of the variable are constant over time. (Ex. Temperature in the weather)
  - **Non-stationary** – when the expected value and/or variance grow over time. (Ex. Stock price)

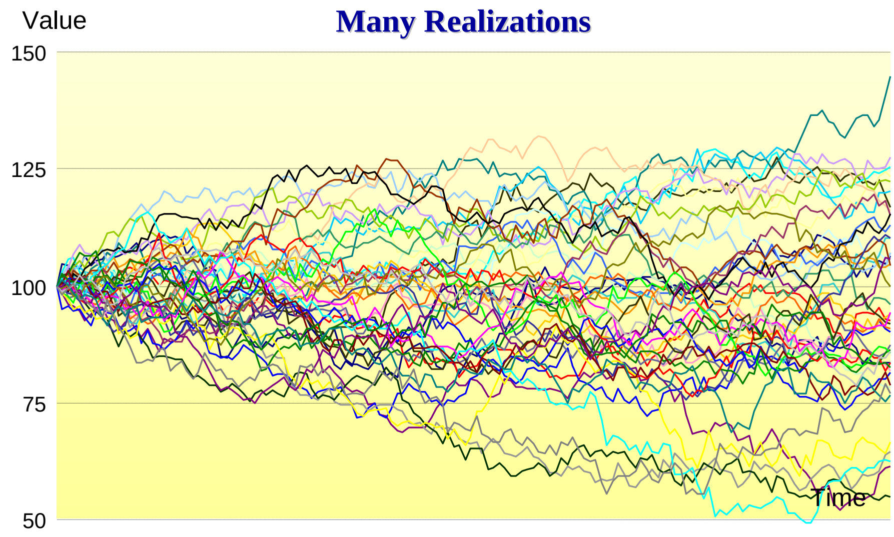

Possible Paths of a Stochastic Variable:
Single Period
Value
105
103
100
98
95
Time


Possible Paths of a Stochastic Variable:
Two Periods
Value
105
103
100
98
95
Time

Possible Paths of a Stochastic Variable:
Multiple Periods, Single Realization



Possible Paths of a Stochastic Variable:
Many Realizations



Stochastic Processes

- Stochastic Processes were first used in physics to describe the motion on particles.
- Stochastic Processes can be classified into the following categories:
  - **Discrete Time Process:** The variable can only change its value at certain fixed moments in time.
  - **Continuous Time Process :** The variable can change its value at any moment in time.
  - **Discrete Variable:** The variable can assume only discrete values
  - **Continuous Variable:** The variable can assume any value within a certain range.

Stochastic Processes

- Most practical problems assume a ***continuous time process*** with ***continuous variables***.
- **Continuous time processes** require the use of calculus to solve the **differential equations** that model these processes.
- Continuous time processes can be **approximated with discrete processes**, which are easier to model.
- We will study the continuous time models and then show the corresponding discrete time processes

Markov Process

- A **Markov Process** is a stochastic process where only the current value of a variable is relevant to forecast the future path of the process.
- This means that historic values and the path by which the variable arrived at its current value are irrelevant to predict its future values.
- It is assumed that stock prices and many other asset prices follow a Markov process.
- Under this assumption, the current price of a stock already reflects all the past information and expectations about the future price of this stock.

Random Walk

- Random Walk is one of the most basic stochastic processes.
- The name come from the path followed by a drunk sailor walking down the pier. His steps swing randomly from right to left and his destination becomes more uncertain with time.
- Random Walk is a discrete time Markov process that has independent increments in the form:

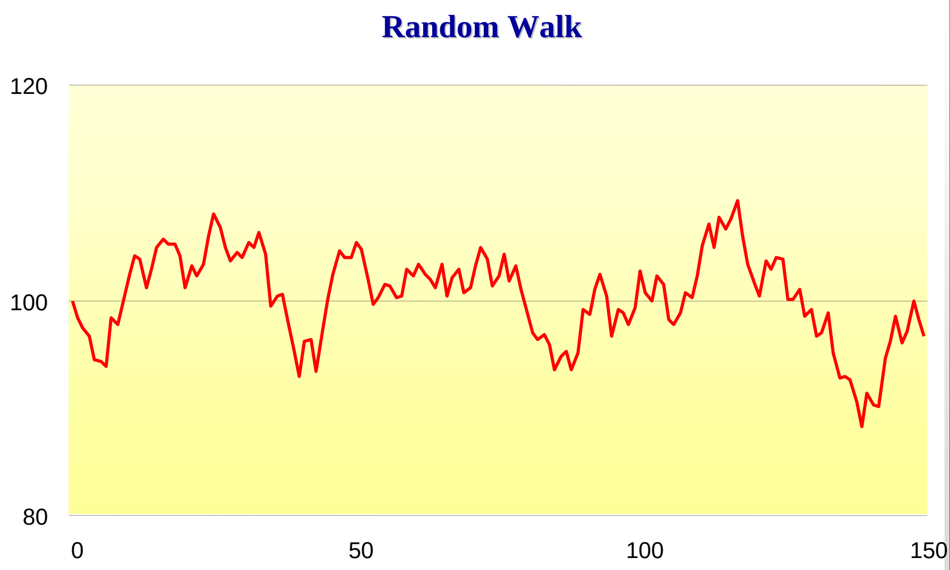
$$S_t = S_{t-1} + \varepsilon_t$$

where S_t is the value of the variable in time *t*

S_{t+1} is the value of the variable in time *t-1*

ε_t is a random variable with probability

$P(\varepsilon_t=1) = P(\varepsilon_t=-1) = 0.5$



Random Walk

- Random walks can also include a drift term, which represents a long term growth rate.
- Without a drift term, the best estimate for the next value of the variable (St+1) is the current value, since the error term is normally distributed with mean zero.
- With a drift term, future values of the variables are expected to increase proportionately to this growth rate.

Wiener Process

- A Wiener Process is a stochastic process that has a mean of zero and a variance of one per year.
- A Wiener Process is a particular case of a Markov process, and is also known as Brownian Motion.
- This process, first described by botanist Robert Brown in 1827, is used in physics to describe the motion of small particles that are subject to a great number of small shocks.
- In 1923 Norbert Wiener developed the mathematical theory of Brownian motion.

Wiener Process

- A Wiener Process has three important characteristics:
  1. It is a continuous time Markov process
  2. Each increment is independent of previous increments
  3. Changes in the process are normally distributed with a variance that increases linearly with time.
- A Wiener Process is a continuous time version of a Random Walk in the form:

$$S_t = S_{t-1} + dz \qquad where\ dz = \varepsilon\sqrt{dt} \quad and\ \varepsilon \sim N(0,1)$$

$$E[dz] = 0 \qquad \text{var}[dz] = dt$$

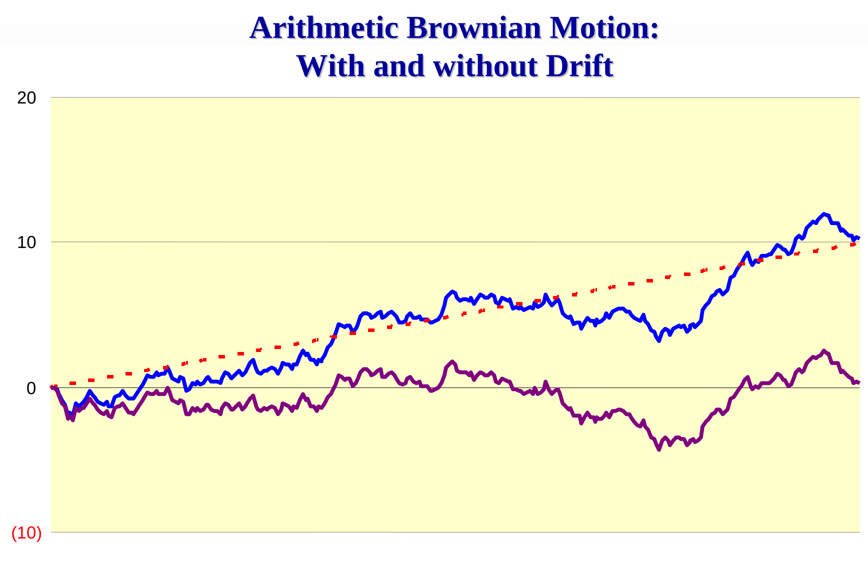
Arithmetic Brownian Motion (ABM)

- A Wiener Process is a stationary process, as it has no drift term.
- If we add a drift term to the Wiener Process we get an Arithmetic Brownian Motion, which has the form:

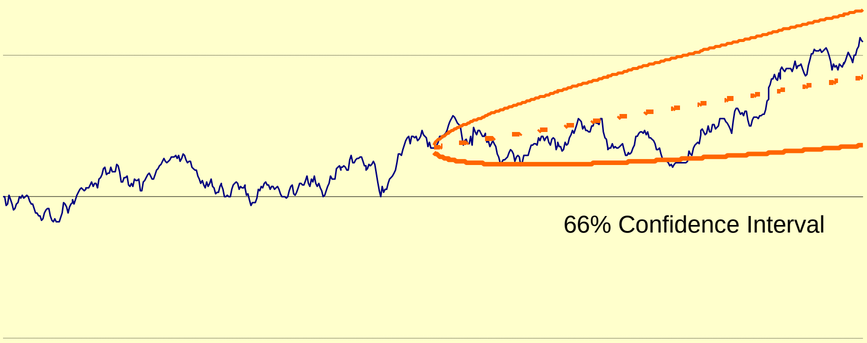
$$S_t = S_{t-1} + \mu dt + \sigma dz$$

$$dS = \mu dt + \sigma dz \qquad dS \sim N(\mu dt, \sigma^2 dt)$$

- The focus is on the change in the value of the variable rather than on the value itself.
- The ABM is also a random walk, and thus is normally distributed.



ABM: Forecast and Confidence Interval



Limitations of the ABM model

- ABM is also called the Additive Model because the variable grows by a constant expected value in each period.
- Since the random term is a normally distributed random variable, the value of the variable can eventually become negative.
- Since we are mainly interested in modeling prices and values of assets, this is a problem, as prices cannot be negative.
- Also, the standard deviation should also be proportional to the price, and not constant as in the ABM model.

Limitations of the ABM model

- For a non dividend paying stock, the ABM model implies that the *rate* of return decreases as the value of the stock increases with time.
- In reality, we know that investors require a constant rate of return, independent of the actual stock price.
- For these reasons, the ABM model is not appropriate for modeling stock prices, or asset values in general.
- A better model for stock prices is one that leads to a constant expected rate of return.

Geometric Brownian Motion (GBM)

- A better model for stock prices is a process where the *proportional* return and volatility are constant.
- This is also referred to as a **Geometric Brownian Motion (GBM),** or **Multiplicative Mode**, and is used extensively to model the behavior of stock prices and many other assets.
- The continuous time form is:

$$dS = \mu S dt + \sigma S dz \quad or \quad \frac{dS}{S} = \mu dt + \sigma dz$$

where μ = Expected rate of return
σ = Volatility of Stock Price

Geometric Brownian Motion (GBM)

- Note that the proportional changes in the variable ***S***, dS/S are normally distributed, since $\frac{dS}{S} = \mu dt + \sigma dz$ is a ABM.
- dS/S is the return on ***S***.
- In continuous time, if $\tilde{S}_t = S_0 e^{\tilde{v}t}$ then v is the return on ***S***.
- For $t = 1$ we have $\tilde{S}_1 = S_0 e^{\tilde{v}}$ and $e^{\tilde{v}} = \tilde{S}_1 / S_0$
- Taking the logarithm, we get $\tilde{v} = \ln\left(\tilde{S}_1 / S_0\right)$

Geometric Brownian Motion (GBM)

- We can see that $\ln\left(\tilde{S}_1 / S_0\right)$ is the return on ***S*** and that this return in normally distributed.
- Since the returns of ***S*** are normally distributed, then ***S*** is *lognormally* distributed. This is an important and very useful characteristic of the GBM.
- The GBM has then three characteristics that make it extremely suitable to model asset prices:
  - It allows for exponential growth, similar to compounded interest.
  - Proportional changes in values are normally distributed, which is easy to manipulate mathematically.
  - The value of the variable cannot become negative, similar to what occurs with prices.

Relations for a GBM process

- S_t:

$$E\left[\tilde{S}_t\right] = S_0 e^{\tilde{\mu} t}$$

$$\operatorname{var}\left[\tilde{S}_t\right] = S_0^2 e^{2\tilde{\mu} t}\left(e^{\sigma^2 t} - 1\right)$$

- Ln S_t:

$$E\left[\ln\frac{\tilde{S}_t}{S_0}\right] = E\left[\tilde{v}\right] = vt$$

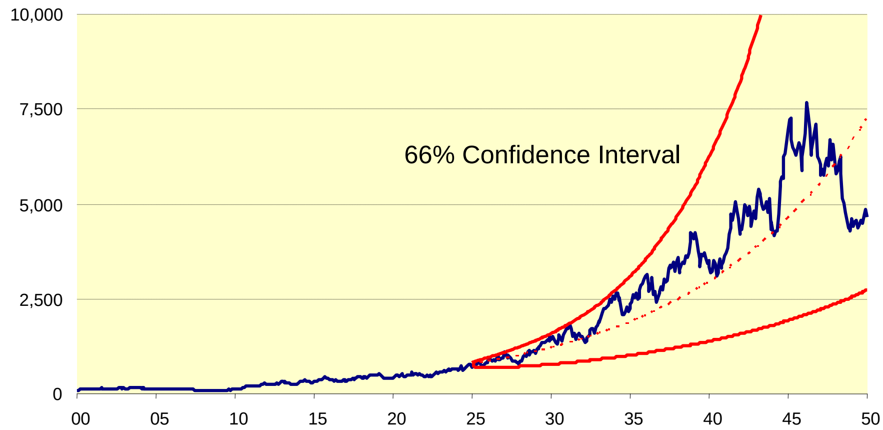
$$\operatorname{var}\left[\ln\frac{\tilde{S}_t}{S_0}\right] = \operatorname{var}\left[\tilde{v}\right] = \sigma^2 t$$

$$v = \mu - \frac{\sigma^2}{2}$$

Geometric Brownian Motion (GBM)

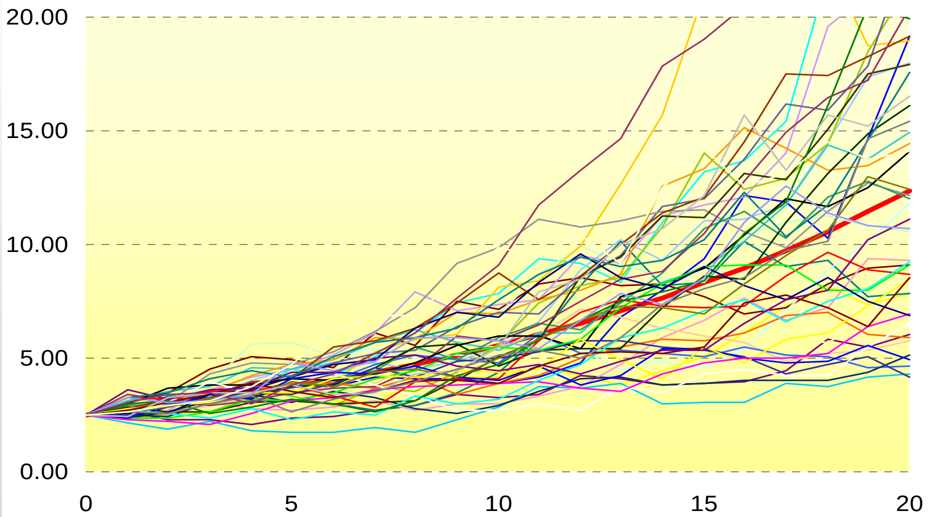
$$dS = \mu S dt + \sigma S dz$$

Geometric Brownian Motion: Forecast and Confidence Interval



Possible paths of a GBM Process

$dS = 0{,}08\ Sdt + 0{,}114\ Sdz$



Mean Reverting Process

- As we saw before, Brownian Motion processes tend to wander far from their starting points.
- While if may be realistic to model the value of most assets in this way, there are a some assets that do not seem to behave in this fashion.
- For example, prices of assets such as oil, copper and other comodities are related to their long term marginal production costs, even though they might fluctuate randomly in the short run.
- As prices vary, producers will increase production in order to take advantage of higher prices or shut down facilities to avoid losses when prices are low
- This will tend to force prices to revert to their long term mean.

Mean Reverting Process

- The simplest form of a Mean Reverting Process is the Onstein – Uhlenbeck process.

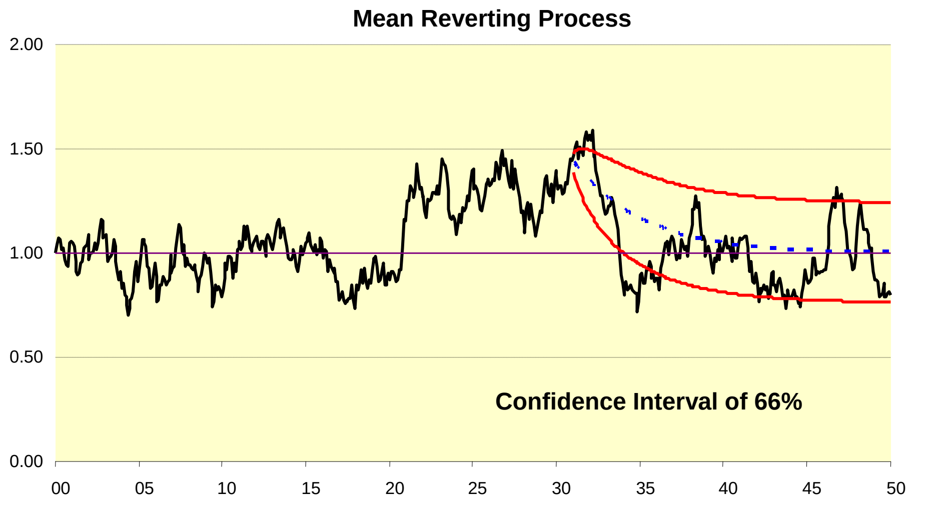
$$\Delta x = \eta(\overline{x} - x)\Delta t + \sigma \Delta z$$

where $\acute{\eta}$ = the speed of reversion

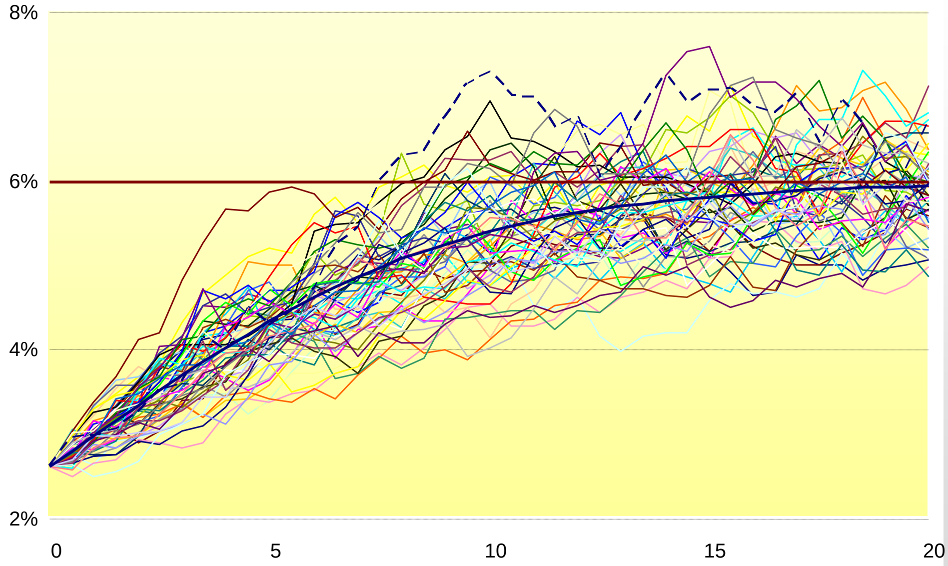
$\overline{x}$ = the long term average to which the process reverts to.

- The speed of reversion indicates how quickly the values revert to the long term mean.

Example: Mean Reverting



Paths of a Mean Reverting Process



Final Comments

- The ABM, GBM and Mean Reversion models are also called **“diffusion” models** because the value of the variable changes in very small increments.
- Processes where the value changes abruptly are called **“jump” models.**
- **Arithmetic Brownian Motion** is more appropriate for physical processes, while **Geometric Brownian Motion** is widely used to model the value of financial and real assets. This will be the main process we will use in this course.
- **Mean Reversion models** are used to model interest rates and commodity prices.
- For **simple option modeling**, we will assume that project values follow a Geometric Brownian Motion (GBM), which has a lognormal distribution.