8-2 Elastic Beam theory

- To derive the DE, we look at an initially straight beam that is elastically deformed by loads applied perpendicular to beam's x-axis & lying in x-v plane of symmetry , Fig 8.7(a)
- Due to loading, the beam deforms under shear & bending
- If beam L >> d, greatest deformation will be caused by bending
- When M deforms the element of beam, the angle between the cross sections becomes dθ, Fig 8.7(b)

8-2 Elastic Beam theory

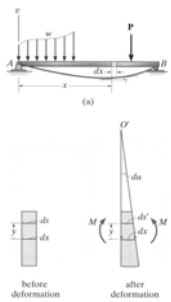


Fig. 8–7

8-2 Elastic Beam theory

- The arc dx that rep a portion of the elastic curve intersects the neutral axis
- The radius of curvature for this arc is defined as the distance, ρ, which is measured from ctr of curvature O' to dx
- Any arc on the element other than dx is subjected to normal strain
- The strain in arc ds located at position y from the neutral axis is

$$\varepsilon = (ds' - ds)/ds$$

8-2 Elastic Beam theory

$$ds = dx = \rho d\theta \text{ and } ds' = (\rho - y)d\theta$$

$$\varepsilon = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} \Rightarrow \frac{1}{\rho} = -\frac{\varepsilon}{y}$$

- If the material is homogeneous & behaves in a linear manner, then Hooke's law applies

$$\varepsilon = \sigma / E$$

- The flexure formula also applies

$$\sigma = -My/I$$

8-2 Elastic Beam theory

- Combining these eqns, we have:

$$\frac{1}{\rho} = \frac{M}{EI} \qquad \text{eqn } 8.1$$

$\rho =$ the radius of curvature at a specific point on the elastic curve

$M =$ internal moment in the beam at the point where ρ is to be determined

$E =$ the material's modulus of elasticity

$I =$ the beam's moment of inertia computed about the neutral axis

8-2 Elastic Beam theory

$EI =$ flexural rigidity; $\quad dx = \rho d\theta$

From eqn 8.1

$$d\theta = \frac{M}{EI} dx \qquad \text{eqn } 8.2$$

$$\text{v} - \text{axis as} + \text{ve} \uparrow, \quad \frac{1}{\rho} = \frac{d^2 v / dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

$$\text{Therefore,} \quad \frac{M}{EI} = \frac{d^2 v / dx^2}{[1 + (dv/dx)^2]^{3/2}} \qquad \text{eqn } 8.3$$

8-2 Elastic Beam theory

- This eqn rep a non-linear second DE
- V=f(x) gives the exact shape of the elastic curve
- The slope of the elastic curve for most structures is very small
- Using small deflection theory, we assume dv/dx ~ 0
- Eqn 3 reduces to

$$\frac{d^2 v}{dx^2} = \frac{M}{EI} \qquad \text{eqn } 8.4$$

8-2 Elastic Beam theory

- By assuming dv/dv ~ 0 $\Rightarrow$ ds in Fig .7(b) is approximately equal to dx

$$ds = 2\overline{dx^2 + dv^2} = 2\overline{1 + (dv/dx)^2} dx \approx dx$$

- This implies that points on the elastic curve will only be displaced vertically & not horizontally

8-3 The double integration method

- M = f(x), successive integration of eqn 8.4 will yield the beam's slope
  - $\theta \approx \tan\theta = dv/dx = \int M/EI\, dx$
- Eqn of elastic curve
  - $V = f(x) = \int M/EI\, dx$
- Consider the beam shown in Fig 8.8
- The internal moment in regions AB, BC & CD must be written in terms of x_1, x_2 and x_3

8-3 The double integration method

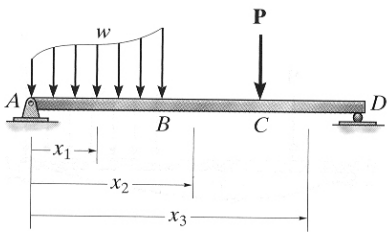
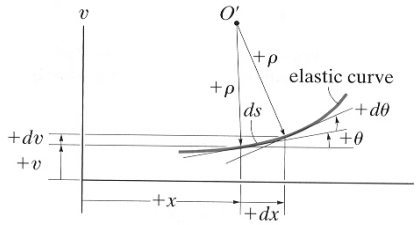


Fig. 8–8

8-3 The double integration method

- Once these functions are integrated through the applications of eqn 8.4 & the constants determined, the functions will give the slope & deflection for each region of the beam
- When applying eqn 8.4, it is important to use the proper sign for M as established by the sign convention used in derivation
- +ve v is upward, hence, the +ve slope angle, θ will be measured counterclockwise from the x-axis
- Reason for this is shown in Fig 8.9(b)

8-3 The double integration method



(b)

Fig. 8–9

8-3 The double integration method

- The constants of integration are determined by evaluating the functions for slope or displacement at a particular point on the beam where the value of the function is known
- These values are called boundary conditions
- Consider the beam shown in Fig 8.10
- Here x_1 and x_2 coordinates are valid within the regions AB & BC

8-3 The double integration method

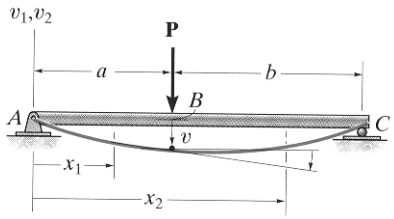


Fig. 8–10

8-3 The double integration method

- Once the functions for the slope & deflections are obtained, they must give the same values for slope & deflection at point B
- This is so as for the beam to be physically continuous

Example 8.3

- The cantilevered beam shown in Fig 8.11(a) is subjected to a couple moment M_o at its end
- Determine the eqn of the elastic curve
- EI is constant

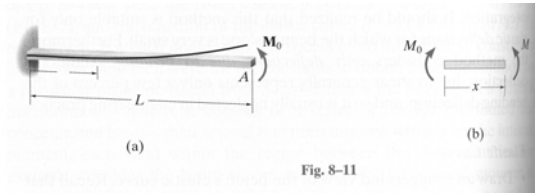


Fig. 8–11

Example 8.3 - solution

- The load tends to deflect the beam as shown in Fig 8.9(a)
- By inspection, the internal moment can be represented throughout the beam using a single x coordinate
- From the free-body diagram, with M acting in +ve direction, Fig 8.11(b), we have:

$$M = M_o$$

Example 8.3 - solution

- Applying eqn 8.4 & integrating twice yields:

$$EI\frac{d^2v}{dx^2} = M_o$$

$$EI\frac{dv}{dx} = M_o x + C_1$$

$$EI\,v = \frac{M_o x^2}{2} + C_1 x + C_2$$

- Using boundary conditions, dv/dx = 0 at x = 0 & v = 0 at x = 0 then $C_1 = C_2 = 0$

Example 8.3 - solution

- Substituting these values into earlier eqns, we get:

with $\theta = dv/dx$

$$\theta = \frac{M_o x}{EI}; \quad v = \frac{M_o x^2}{2EI}$$

- Max slope & disp occur at A (x = L) for which

$$\theta_A = \frac{M_o L}{EI}; \quad v_A = \frac{M_o L^2}{2EI}$$

Example 8.3 - solution

- The +ve result for θ_A indicates counterclockwise rotation & the +ve result for v_A indicates that it is upwards
- This agrees with results sketched in Fig 8.11(a)
- In order to obtain some idea to the actual magnitude of the slope
- Consider the beam in Fig 8.11(a) to:
  - Have a length of 3.6m
  - Support a couple moment of 20kNm
  - Be made of steel having E_{st} = 200GPa

Example 8.3 - solution

- If this beam were designed w/o a fos by assuming the allowable normal stress = yield stress = 250kNm/^{2}
- A W6 x 9 would be found to be adequate

$$\theta_A = \frac{20kNm(3.6m)}{[200(10^6)kN/m_2][6.8(10^6)(10^{-12})m^4} = 0.0529rad$$

$$v_A = \frac{20kNm(3.6m)^2}{2[200(10^6)kN/m_2][6.8(10^6)(10^{-12})m^4} = 95.3mm$$

Example 8.3 - solution

- Since θ^2_A = 0.00280(10-^{6})<<1, this justifies the use of eqn 8.4
- Also this numerical application is for cantilevered beam, we have obtained larger values for max θ and v than would have been obtained if the beam were supported using pins, rollers or other supports

Example 8.3 - solution