To derive the DE, we look at an initially straight beam that is elastically deformed by loads applied perpendicular to beam’s x-axis & lying in x-y plane of symmetry, Fig 8.7(a).

Due to loading, the beam deforms under shear & bending.

If beam L >> d, greatest deformation will be caused by bending.

When M deforms the element of beam, the angle between the cross sections becomes $d\theta$, Fig 8.7(b).

The arc $dx$ that rep a portion of the elastic curve intersects the neutral axis.

The radius of curvature for this arc is defined as the distance, $\rho$, which is measured from ctr of curvature $O'$ to $dx$.

Any arc on the element other than $dx$ is subjected to normal strain.

The strain in arc $ds$ located at position $y$ from the neutral axis is $$\varepsilon = \frac{(ds'-ds)}{ds}$$

If the material is homogeneous & behaves in a linear manner, then Hooke’s law applies

$$\varepsilon = \frac{\sigma}{E}$$

The flexure formula also applies

$$\sigma = -\frac{My}{I}$$
Combining these eqns, we have:

\[
\frac{1}{\rho} = \frac{M}{EI} \quad \text{eqn 8.1}
\]

\(\rho\) = the radius of curvature at a specific point on the elastic curve

\(M\) = internal moment in the beam at the point where \(\rho\) is to be determined

\(E\) = the material's modulus of elasticity

\(I\) = the beam’s moment of inertia computed about the neutral axis

**8.3 eqn**

\[
\begin{align*}
\frac{d^2v}{dx^2} &= \frac{M}{EI} \\
E &= \frac{M}{EI} \\
\end{align*}
\]

**8.4 eqn**

\[
\begin{align*}
\frac{d^2v}{dx^2} &= \frac{M}{EI} \\
\end{align*}
\]

This eqn rep a non-linear second DE

\(V = f(x)\) gives the exact shape of the elastic curve

The slope of the elastic curve for most structures is very small

Using small deflection theory, we assume \(dv/dx \approx 0\)

Eqn 3 reduces to

\[
\frac{d^2v}{dx^2} = \frac{M}{EI}
\]

**8.2 Elastic Beam theory**

\(EI\) = flexural rigidity; \(dx = \rho d\theta\)

From eqn 8.1

\[
d\theta = \frac{M}{EI} \cdot dx \quad \text{eqn 8.2}
\]

\(v\) – axis as +ve \(\hat{\imath}\),

\[
\frac{1}{\rho} = \frac{d^2v}{dx^2} \quad \text{eqn 8.3}
\]

Therefore, \(\frac{M}{EI} = \frac{d^2v}{dx^2} \quad \text{eqn 8.3}\)

By assuming \(dv/dv \approx 0\) \(\Rightarrow ds\) in Fig. 7(b) is approximately equal to \(dx\)

\[
ds = 2dx^2 + dv^2 = 2(\frac{dv}{dx})^2dx \approx dx
\]

This implies that points on the elastic curve will only be displaced vertically & not horizontally.
8-3 The double integration method

- $M = f(x)$, successive integration of eqn 8.4 will yield the beam’s slope
  - $\theta \approx \tan \theta = \frac{dv}{dx} = \int \frac{M}{EI} \, dx$
- Eqn of elastic curve
  - $V = f(x) = \int \frac{M}{EI} \, dx$
- Consider the beam shown in Fig 8.8
- The internal moment in regions AB, BC & CD must be written in terms of $x_1$, $x_2$ and $x_3$

Once these functions are integrated through the applications of eqn 8.4 & the constants determined, the functions will give the slope & deflection for each region of the beam

- When applying eqn 8.4, it is important to use the proper sign for $M$ as established by the sign convention used in derivation
  - $+ve$ $v$ is upward, hence, the $+ve$ slope angle, $\theta$ will be measured counterclockwise from the $x$-axis
  - Reason for this is shown in Fig 8.9(b)

Fig. 8-8

Fig. 8-9
8-3 The double integration method

- The constants of integration are determined by evaluating the functions for slope or displacement at a particular point on the beam where the value of the function is known.
- These values are called boundary conditions.
- Consider the beam shown in Fig 8.10.
- Here $x_1$ and $x_2$ coordinates are valid within the regions AB & BC.

Once the functions for the slope & deflections are obtained, they must give the same values for slope & deflection at point B.

This is so as for the beam to be physically continuous.

Example 8.3

- The cantilevered beam shown in Fig 8.11(a) is subjected to a couple moment $M_o$ at its end.
- Determine the eqn of the elastic curve.
- $EI$ is constant.
Example 8.3 - solution

- The load tends to deflect the beam as shown in Fig 8.9(a).
- By inspection, the internal moment can be represented throughout the beam using a single x coordinate.
- From the free-body diagram, with M acting in +ve direction, Fig 8.11(b), we have:
  \[ M = M_o \]

Example 8.3 - solution

- Substituting these values into earlier eqns, we get:
  \[ \theta = \frac{dv}{dx} \]
  \[ \theta = \frac{M_o x}{EI}; \quad v = \frac{M_o x^2}{2EI} \]
- Max slope & disp occur at A (x = L) for which
  \[ \theta_A = \frac{M_o L}{EI}; \quad v_A = \frac{M_o L^2}{2EI} \]

Example 8.3 - solution

- Applying eqn 8.4 & integrating twice yields:
  \[ EI \frac{d^2 v}{dx^2} = M_o \]
  \[ EI \frac{dv}{dx} = M_o x + C_1 \]
  \[ EI v = \frac{M_o x^2}{2} + C_1 x + C_2 \]
- Using boundary conditions, \( dv/dx = 0 \) at \( x = 0 \) & \( v = 0 \) at \( x = 0 \) then \( C_1 = C_2 = 0 \)

Example 8.3 - solution

- The +ve result for \( \theta_A \) indicates counterclockwise rotation & the +ve result for \( v_A \) indicates that it is upwards.
- This agrees with results sketched in Fig 8.11(a).
- In order to obtain some idea to the actual magnitude of the slope.
- Consider the beam in Fig 8.11(a) to:
  - Have a length of 3.6m
  - Support a couple moment of 20kNm
  - Be made of steel having \( E_{st} = 200 \text{GPa} \).
Example 8.3 - solution

- If this beam were designed w/o a fos by assuming the allowable normal stress = yield stress = 250kNm/m
- A W6 x 9 would be found to be adequate

\[ \theta_d = \frac{20kNm(3.6m)}{[200(10^9)kN / m][6.8(10^9)(10^{-12})m^3]} = 0.0529 \text{rad} \]

\[ v_d = \frac{20kNm(3.6m)^2}{[2][200(10^9)kN / m][6.8(10^9)(10^{-12})m^3]} = 95.3 \text{mm} \]

Example 8.3 - solution

- Since \( \theta_A = 0.00280(10^{-4}) < 1 \), this justifies the use of eqn 8.4
- Also this numerical application is for cantilevered beam, we have obtained larger values for max \( \theta \) and \( v \) than would have been obtained if the beam were supported using pins, rollers or other supports