Structural Analysis

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6th Edition
in SI Units

Teaching Slides
Chapter 9
Deflections using energy methods

PEARSON Education Asia

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Outline

- External work & strain energy
- Principle of work & energy
- Principle of virtual work
- Method of virtual work: trusses
- Method of virtual work: beams & frames
- Virtual strain energy caused by axial load, shear, torsion & temperature
- Castigliano's theorem
- Castigliano's theorem for trusses
- Castigliano's theorem for beams & frames

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9-1 External Work & Strain Energy

- For more complicated loadings or for structures such as trusses & frames, it is suggested that energy methods be used for the computations
- Most energy methods are based on the conservation of energy principal
- Work done by all external forces acting on a structure, U_e is transformed into internal work or strain energy U_i
  - $U_e = U_i$ eqn 9.1

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9-1 External Work & Strain Energy

- If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed
- When a force F undergoes a disp dx in the same direction as the force, the work done is
  - $d\,U_e = F\,dx$
- If the total disp is x, the work becomes:

$$U_e = \int_0^x F\,dx \qquad \text{eqn 9.2}$$

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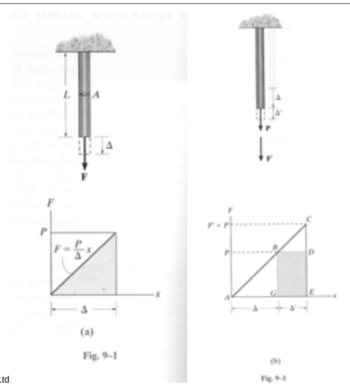
9-1 External Work & Strain Energy

- Consider the effect caused by an axial force applied to the end of a bar as shown in Fig 9.1(a)
- F is gradually increased from 0 to some limiting value F = P
- The final elongation of the bar becomes Δ
- If the material has a linear elastic response, then $F = (P/\Delta)x$

9-1 External Work & Strain Energy

- Fig 9.1



9-1 External Work & Strain Energy

- Substituting into eqn 9.2 & integrating from) to Δ, we get:

$$U_e = \frac{1}{2} P\Delta \qquad \text{eqn 9.3}$$

which is the shaded area under Fig 9.1(a)

- Suppose P is already applied to the bar & that another force F' is now applied, so that the bar deflects further by an amount Δ', Fig 9.1(b)

9-1 External Work & Strain Energy

- The work done by P when the bar undergoes the further deflection is then
  - $d\,U_e' = P\Delta'$ eqn 9.4
- Here the work rep the shaded rectangular area in Fig 9.1(b)
- In this case, P does not change its magnitude since Δ' is caused only by F'
  - Work = force x disp

9-1 External Work & Strain Energy

- When a force P is applied to the bar, followed by an application of the force F’, the total work done by both forces is rep by the triangular area ACE in Fig 9.1(b)
- The triangular area ABG rep the work of P that is caused by disp Δ
- The triangular area BCD rep the work of F’ since this force causes a dsip Δ’
- Lastly the shaded rectangular area BDEG rep the additional work done by P

9-1 External Work & Strain Energy

- The work of a moment = magnitude of the moment (M) x the angle ($d\theta$) through which it rotates, Fig 9.2
  - d U_e = M $d\theta$
- If the total angle of rotation is θ rad, the work becomes

$$Ue = \int_0^\theta M d\theta \qquad \text{eqn 9.5}$$

9-1 External Work & Strain Energy

- Fig 9.2

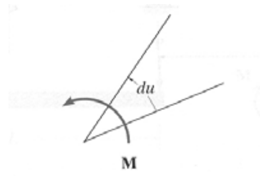


Fig. 9–2

9-1 External Work & Strain Energy

- If the moment is applied gradually to a structure having a linear elastic response from 0 to M, then the work done is

$$Ue = \frac{1}{2} M\theta \qquad \text{eqn 9.6}$$

- However, if the moment is already applied to the structure & other loadings further distort the structure an amount θ’, then M rotates θ’ & the work done is

$$Ue' = M\theta' \qquad \text{eqn 9.7}$$

9-1 External Work & Strain Energy

- When an axial force N is applied gradually to the bar in Fig 9.3, it will strain the material such that the external work done by N will be converted into strain energy
- Provided the material is linearly elastic, Hooke's Law is valid
  - $\sigma = E\varepsilon$
- If the bar has a constant x-sectional area A and length L

9-1 External Work & Strain Energy

- Fig 9.3

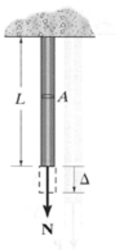


Fig. 9–3

9-1 External Work & Strain Energy

- The normal stress is σ = N/A
- The final strain is $\varepsilon = \Delta/L$
- Consequently, N/A = E(Δ/L)
- Final deflection: $$\Delta = \frac{NL}{AE} \quad \text{eqn 9.8}$$
- Substituting into eqn 9.3 with P = N,

$$U_i = \frac{N^2L}{2AE} \quad \text{eqn 9.9}$$

9-1 External Work & Strain Energy

- Consider the beam shown in Fig 9.4(a)
- P & w are gradually apply
- These loads create an internal moment M in the beam at a section located a distance x from the left support
- The resulting rotation of the DE dx, Fig 9.4(b) can be found from eqn 8.2
- Consequently, the strain energy or work stored in the element is determined from eqn 9.6 since the internal moment is gradually developed

9-1 External Work & Strain Energy

- Hence,

$$dU_i = \frac{M^2 dx}{2EI} \quad \text{eqn 9.10}$$

- The strain energy for the beam is determined by integrating this result over the beam's length

$$U_i = \int_0^L \frac{M^2 dx}{2EI} \quad \text{eqn 9.11}$$

9-1 External Work & Strain Energy

- Fig 9.4

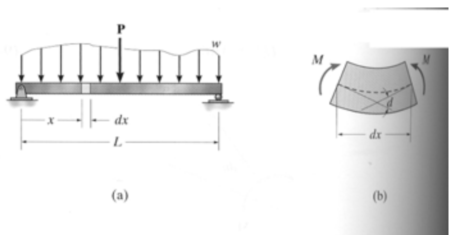


Fig. 9–4

9-2 Principle of Work & Energy

- Consider finding the disp at a point where the force P is applied to the cantilever beam in Fig 9.5
- From eqn 9.3, the external work:

$$U_e = \frac{1}{2} P\Delta$$

- To obtain the resulting strain energy, we must first determine the internal moment as a function of position x in the beam & apply eqn 9.11

9-2 Principle of Work & Energy

- In this case, M = - Px so that:

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{1}{6}\frac{P^2 L^3}{EI}$$

- Equating the ext work to int strain energy & solving for the unknown disp, we have:

$$U_e = U_i$$
$$\frac{1}{2} P\Delta = \frac{1}{6}\frac{P^2 L^3}{EI}$$
$$\Delta = \frac{PL^3}{3EI}$$

9-2 Principle of Work & Energy

- Limitations
  - It will be noted that only one load may be applied to the structure
  - Only the disp under the force can be obtained

9-3 Principle of virtual work

- If we take a deformable structure of any shape or size & apply a series of external loads P to it, it will cause internal loads u at points throughout the structure
- It is necessary that the external & internal loads be related by the eqn of equilibrium
- As a consequence of these loadings, external disp Δ will occur at the P loads & internal disp δ will occur at each point of internal loads u
- In general, these disp do not have to be elastic, & they may not be related to the loads

9-3 Principle of virtual work

- In general, the principle states that:

$$\underbrace{\sum P\Delta}_{\text{Work of Ext loads}} = \underbrace{\sum u\delta}_{\text{Work of Int loads}}$$

- Consider the structure (or body) to be of arbitrary shape as shown in Fig 9.6(b)
- Suppose it is necessary to determine the disp Δ of point A on the body caused by the "real loads" P_1, P_2 and P_3

9-3 Principle of virtual work

- It is to be understood that these loads cause no movement of the supports
- They can strain the material beyond the elastic limit
- Since no external load acts on the body at A and in the direction of Δ, the disp Δ, the disp can be determined by first placing on the body a "virtual" load such that this force P' acts in the same direction as Δ, Fig 9.6(a)

9-3 Principle of virtual work

- Fig 9.6

9-3 Principle of virtual work

- We will choose P' to have a unit magnitude, P' =1
- Once the virtual loadings are applied, then the body is subjected to the real loads P_1, P_2 and P_3, Fig 9.6(b)
- Point A will be displaced an amount Δ causing the element to deform an amount dL

9-3 Principle of virtual work

- As a result, the external virtual force P' & internal load u "ride along" by Δ and dL & therefore, perform external virtual work of 1. Δ on the body and internal virtual work of u.dL on the element

$$1.\Delta = \sum u.dL \quad \text{eqn 9.13}$$

- By choosing P' = 1, it can be seen from the solution for Δ follows directly since $\Delta = \Sigma udL$
- A virtual couple moment M' having a unit magnitude is applied at this point

9-3 Principle of virtual work

- This couple moment causes a virtual load u_θ in one of the elements of the body
- Assuming that the real loads deform the element an amount dL, the rotation θ can be found from the virtual –work eqn

$$1.\theta = \sum u_\theta .dL \quad \text{eqn 9.14}$$