Modal equations for undamped systems

We have derive the equations of motion for MDOF system as

$$m\ddot{u} + ku = p(t)$$

These differential equations are coupled when one of the mass or stiffness matrices is not diagonal. Then, all equations must be solved simultaneously, which is difficult to carry out.

The problem can be solved easier if we convert the equations in term of displacements $u(t)$ into equations in terms modal coordinates $q(t)$. The displacement vector $u(t)$ can be expressed as

$$u(t) = \sum_{r=1}^{N} \phi_r(q_r(t) = \Phi q(t)$$

Substitute this in the equation of motion

$$\sum_{r=1}^{N} m\phi_r\ddot{q}_r(t) + \sum_{r=1}^{N} k\phi_rq_r(t) = p(t)$$

Pre-multiply by $\phi_n^T$

$$\sum_{r=1}^{N} \phi_n^T m\phi_r\ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T k\phi_rq_r(t) = \phi_n^T p(t)$$

Because of the orthogonality properties of modes, the only nonzero term in the summation is when $r = n$, so

$$\phi_n^T m\phi_n\ddot{q}_n(t) + \phi_n^T k\phi_nq_n(t) = \phi_n^T p(t)$$

or

$$M_n\ddot{q}_n(t) + K_nq_n(t) = P_n(t)$$
where

\[ M_n = \phi_n^T m \phi_n \quad K_n = \phi_n^T k \phi_n \quad P_n(t) = \phi_n^T p(t) \]

\( M_n \) is called generalized mass for the nth natural mode
\( K_n \) is called generalized stiffness for the nth mode and
\( P_n(t) \) is called generalized force for the nth mode

Divide the equation by \( M_n \). The equation that governs the nth modal coordinate \( q_n(t) \) for the nth mode becomes

\[ \ddot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n} \]

\( q_n(t) \) is the only unknown in this equation and the solution can be obtained as for the response of a SDOF system. The modal coordinate for all modes can be obtained from such equation \((n = 1, 2, \ldots, N)\). The matrix form of all equations for \( n = 1, 2, \ldots, N \) is

\[ M\ddot{q} + Kq = P(t) \]

where \( M \) and \( K \) are diagonal matrices consisting of \( M_n \) and \( K_n \), respectively, on the main diagonal. Recall that

\[ M = \Phi^T m \Phi \quad K = \Phi^T k \Phi \]

![Figure 12.3.1](image) Generalized SDF system for the nth natural mode.
Example 12.2

Consider the systems and excitation of Example 12.1. By modal analysis determine the steady-state response of the system.

Solution The natural vibration frequencies and modes of this system were determined in Example 10.4, from which the generalized masses and stiffnesses are calculated using Eq. (12.3.4). These results are summarized next:

\[
\omega_1 = \sqrt{\frac{k}{2m}} \quad \omega_2 = \sqrt{\frac{2k}{m}}
\]

\[
\phi_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}^T \quad \phi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T
\]

\[
M_1 = \frac{3m}{2} \quad M_2 = 3m
\]

\[
K_1 = \frac{3k}{4} \quad K_2 = 6k
\]

1. Compute the generalized forces.

\[
P_1(t) = \phi_1^T \mathbf{p}(t) = \left(\frac{p_o}{2}\right) \sin \omega t \\
P_2(t) = \phi_2^T \mathbf{p}(t) = -\frac{p_o}{2} \sin \omega t
\]

2. Set up the modal equations.

\[
M_n \ddot{q}_n + K_n q_n = P_{no} \sin \omega t
\]
3. Solve the modal equations. To solve Eq. (b) we draw upon the solution presented in
Eq. (3.1.7) for a SDF system subjected to harmonic force. The governing equation is

\[ m \ddot{u} + ku = p_o \sin \omega t \]  

and its steady-state solution is

\[ u(t) = \frac{p_o}{k} C \sin \omega t \quad C = \frac{1}{1 - (\omega/\omega_n)^2} \]  

where \( \omega_n = \sqrt{k/m} \). Comparing Eqs. (c) and (b), the solution for Eq. (b) is

\[ q_n(t) = \frac{P_{no}}{K_n} C_n \sin \omega t \]  

where \( C_n \) is given by Eq. (d) with \( \omega_n \) interpreted as the natural frequency of the \( n \)th mode. Substituting for \( P_{no} \) and \( K_n \) for \( n = 1 \) and \( 2 \) gives

\[ q_1(t) = \frac{2p_o}{3k} C_1 \sin \omega t \quad q_2(t) = -\frac{p_o}{6k} C_2 \sin \omega t \]  

4. Determine the modal responses. The \( n \)th mode contribution to displacements—from
Eq. (12.3.2)—is \( u_n(t) = \phi_n q_n(t) \). Substituting Eq. (f) gives the displacement response due to
the two modes:

\[ u_1(t) = \phi_1 \frac{2p_o}{3k} C_1 \sin \omega t \quad u_2(t) = \phi_2 \frac{-p_o}{6k} C_2 \sin \omega t \]  

5. Combine the modal responses:

\[ u(t) = u_1(t) + u_2(t) \quad \text{or} \quad u_j(t) = u_{j1}(t) + u_{j2}(t) \quad j = 1, 2 \]  

Substituting Eq. (g) and for \( \phi_1 \) and \( \phi_2 \) gives

\[ u_1(t) = \frac{p_o}{6k} (2C_1 + C_2) \sin \omega t \quad u_2(t) = \frac{p_o}{6k} (4C_1 - C_2) \sin \omega t \]  

These results are equivalent to those obtained in Example 12.1 by solving the coupled equations (12.3.1) of motion.
Modal equations for damped systems

When damping is included, the equations of motion for MDOF system are

\[ \mathbf{m} \ddot{\mathbf{u}} + \mathbf{c} \dot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{p}(t) \]

These are coupled equations. The set of equations could be uncoupled by transforming the equation in term of displacements \( \mathbf{u}(t) \) into equations in terms modal coordinates \( \mathbf{q}(t) \). The displacement vector \( \mathbf{u}(t) \) can be expressed as

\[ \mathbf{u}(t) = \sum_{r=1}^{N} \phi_r \mathbf{q}_r(t) = \Phi \mathbf{q}(t) \]

Substitute this in the equation of motion

\[ \sum_{r=1}^{N} \mathbf{m} \phi_r \ddot{\mathbf{q}}_r(t) + \sum_{r=1}^{N} \mathbf{c} \phi_r \dot{\mathbf{q}}_r(t) + \sum_{r=1}^{N} \mathbf{k} \phi_r \mathbf{q}_r(t) = \mathbf{p}(t) \]

Pre-multiply by \( \phi_n^T \)

\[ \sum_{r=1}^{N} \phi_n^T \mathbf{m} \phi_r \ddot{\mathbf{q}}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{c} \phi_r \dot{\mathbf{q}}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k} \phi_r \mathbf{q}_r(t) = \phi_n^T \mathbf{p}(t) \]

Because of the orthogonality properties of modes, the only nonzero term in the 1st and 3rd summation is when \( r = n \), so

\[ M_n \ddot{\mathbf{q}}_n(t) + \sum_{r=1}^{N} C_{nr} \dot{\mathbf{q}}_r + K_n \mathbf{q}_n(t) = P_n(t) \]

where

\[ C_{nr} = \phi_n^T \mathbf{c} \phi_r \]

The nth equation may still involve \( \dot{\mathbf{q}}_r \) of other modes.

Above is equation for nth modal coordinate. If we put equations for all \( n \) together in matrix, we get

\[ \mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{P}(t) \]
C will be diagonal when the system has classical damping. $C_{nr} = 0$ when $n \neq r$. Then, we will have an uncoupled equation.

$$M_n \ddot{q}_n(t) + C_n \dot{q}_n + K_n q_n(t) = P_n(t)$$

Divide the equation by $M_n$. The equation that governs the $n$th modal coordinate $q_n(t)$ for the $n$th mode becomes

$$\ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$$

$\xi_n$ is the damping ratio for the $n$th mode. The solution of this equation is the response of a damped SDF system.

**Figure 12.4.1** Generalized SDF system for the $n$th natural mode.
Displacement response

Once the modal coordinate \( q_n(t) \) have been determined by solving the modal equations. The contribution of the nth mode to displacement of MDOF system is

\[
\mathbf{u}_n(t) = \phi_n q_n(t)
\]

By superposition the response contribution from all modes, the total displacement is then

\[
\mathbf{u}(t) = \sum_{n=1}^{N} \mathbf{u}_n(t) = \sum_{n=1}^{N} \phi_n q_n(t) = \Phi \mathbf{q}(t)
\]

This method to determine the response of MDF system to excitation is known as the classical modal analysis or classical mode superposition method, or modal analysis.

Modal analysis can be used to solve linear system with classical damping only. Damping must be classical to obtain modal equations that are uncoupled.

Element forces

To determine other response quantities \( r(t) \), the concept of superposition is used

\[
r(t) = \sum_{n=1}^{N} r_n(t)
\]

where \( r_n(t) \) is the contribution of nth mode to that response quantities. It is the response due to an equivalent static force.

\[
f_n(t) = k\mathbf{u}_n(t) = \omega_n^2 \mathbf{m} \phi_n q_n(t)
\]

Note that

\[
\mathbf{f}(t) = k\mathbf{u}(t) = \sum_{n=1}^{N} k\mathbf{u}_n(t) = \sum_{n=1}^{N} f_n(t)
\]
Example 12.3

Consider the systems and excitation of Example 12.1. Determine the spring forces $V_j(t)$ for the system of Fig. 12.1.1a, or story shears $V_j(t)$ in the system of Fig. 12.1.1b, without introducing equivalent static forces. Consider only the steady-state response.

**Solution**  Steps 1, 2, 3a, and 3b of the analysis summary of Section 12.7 have already been completed in Example 12.2.

**Step 3c:** The spring forces in the system of Fig. 12.1.1a or the story shears in the system of Fig. 12.1.1b are

\[ V_{1n}(t) = k_1 u_{1n}(t) = k_1 \phi_{1n} q_n(t) \quad (a) \]
\[ V_{2n}(t) = k_2 [u_{2n}(t) - u_{1n}(t)] = k_2 (\phi_{2n} - \phi_{1n}) q_n(t) \quad (b) \]

Substituting Eq. (f) of Example 12.2 in Eqs. (a) and (b) with \( n = 1, k_1 = 2k, k_2 = k, \phi_{11} = \frac{1}{2}, \) and \( \phi_{21} = 1 \) gives the forces due to the first mode:

\[ V_{11}(t) = \frac{2p_o}{3} C_1 \sin \omega t \quad V_{21}(t) = \frac{p_o}{3} C_1 \sin \omega t \quad (c) \]

Substituting Eq. (f) of Example 12.2 in Eqs. (a) and (b) with \( n = 2, \phi_{12} = -1, \) and \( \phi_{22} = 1 \) gives the second-mode forces:

\[ V_{12}(t) = \frac{p_o}{3} C_2 \sin \omega t \quad V_{22}(t) = -\frac{p_o}{3} C_2 \sin \omega t \quad (d) \]

**Step 4b:** Substituting Eqs. (c) and (d) in $V_j(t) = V_{j1}(t) + V_{j2}(t)$ gives

\[ V_1(t) = \frac{p_o}{3} (2C_1 + C_2) \sin \omega t \quad V_2(t) = \frac{p_o}{3} (C_1 - C_2) \sin \omega t \quad (e) \]

Equation (e) gives the time variation of spring forces and story shears. For a given $p_o$ and $\omega$ and the $\omega_n$ already determined, all quantities on the right side of these equations are known; thus $V_j(t)$ can be computed.
Example 12.4

Repeat Example 12.3 using equivalent static forces.

Solution  From Eq. (12.6.2), for a lumped-mass system the equivalent static force in the $j$th DOF due to the $n$th mode is

$$f_{jn}(t) = \omega_n^2 m_j \phi_{jn} q_n(t) \quad (a)$$

Step 3c: In Eq. (a) with $n = 1$, substitute $m_1 = 2m$, $m_2 = m$, $\phi_{11} = \frac{1}{2}$, $\phi_{21} = 1$, $\omega_1^2 = k/2m$, and $q_1(t)$ from Eq. (f) of Example 12.2 to obtain

$$f_{11}(t) = \frac{P_o}{3} C_1 \sin \omega t \quad f_{21}(t) = \frac{P_o}{3} C_1 \sin \omega t \quad (b)$$

In Eq. (a) with $n = 2$, substituting $m_1 = 2m$, $m_2 = m$, $\phi_{12} = -1$, $\phi_{22} = 1$, $\omega_2^2 = 2k/m$, and $q_2(t)$ from Eq. (f) of Example 12.2 gives

$$f_{12}(t) = \frac{2P_o}{3} C_2 \sin \omega t \quad f_{22}(t) = -\frac{P_o}{3} C_2 \sin \omega t \quad (c)$$

Static analysis of the systems of Fig. E12.4 subjected to forces $f_{jn}(t)$ gives the two spring forces and story shears due to the $n$th mode:

$$V_{1n}(t) = f_{1n}(t) + f_{2n}(t) \quad V_{2n}(t) = f_{2n}(t) \quad (d)$$

Substituting Eq. (b) in Eq. (d) with $n = 1$ gives the first mode forces that are identical to Eq. (c) of Example 12.3. Similarly, substituting Eq. (c) in Eq. (d) with $n = 2$ gives the second-mode results that are identical to Eq. (d) of Example 12.3.

Step 4: Proceed as in step 4b of Example 12.3.
Example 12.5

Consider the system and excitation of Example 12.1 with modal damping ratios $\zeta_n$. Determine the steady-state displacement amplitudes of the system.

**Solution** Steps 1 and 2 of the analysis summary have been completed in Example 12.2.

**Step 3:** The modal equations without damping were developed in Example 12.2. Now including damping they become

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_{n0} \sin \omega t$$

(a)

where $M_n$, $K_n$, and $P_{n0}$ are available and $C_n$ is known in terms of $\zeta_n$.

To solve Eq. (a), we draw upon the solution presented in Eq. (3.2.3) for an SDF system with damping subjected to harmonic force. The governing equation is

$$miu + cu + ku = p_0 \sin \omega t$$

(b)

and its steady-state solution is

$$u(t) = \frac{p_0}{k} (C \sin \omega t + D \cos \omega t)$$

(c)

with

$$C = \frac{1 - (\omega/\omega_n)^2}{\left(1 - (\omega/\omega_n)^2\right)^2 + (2\zeta \omega/\omega_n)^2}$$

$$D = \frac{-2\zeta \omega/\omega_n}{\left(1 - (\omega/\omega_n)^2\right)^2 + (2\zeta \omega/\omega_n)^2}$$

(d)

where $\omega_n = \sqrt{k/m}$ and $\zeta = c/(2m)\omega_n$.

Comparing Eqs. (b) and (a), the solution for the latter is

$$q_n(t) = \frac{P_{n0}}{K_n} (C_n \sin \omega t + D_n \cos \omega t)$$

(e)

where $C_n$ and $D_n$ are given by Eq. (d) with $\omega_n$ interpreted as the natural frequency of the $n$th mode and $\zeta = \zeta_n$, the damping ratio for the $n$th mode. Substituting for $P_{n0}$ and $K_n$ for $n = 1$ and 2 gives

$$q_1(t) = \frac{2p_0}{3k} \left(C_1 \sin \omega t + D_1 \cos \omega t\right)$$

(f)

$$q_2(t) = -\frac{p_0}{6k} \left(C_2 \sin \omega t + D_2 \cos \omega t\right)$$

(g)

**Steps 3b and 4:** Substituting $q_n(t)$ in Eqs. (12.5.2) gives the nodal displacements:

$$u_1(t) = \frac{1}{2} q_1(t) - q_2(t) \quad u_2(t) = q_1(t) + q_2(t)$$

Substituting Eqs. (f) and (g) for $q_n(t)$ gives

$$u_1(t) = \frac{p_0}{6k} \left[(2C_1 + C_2) \sin \omega t + (2D_1 + D_2) \cos \omega t\right]$$

(h)

$$u_2(t) = \frac{p_0}{6k} \left[(4C_1 - C_2) \sin \omega t + (4D_1 - D_2) \cos \omega t\right]$$

(i)

The displacement amplitudes are

$$u_{10} = \frac{p_0}{6k} \sqrt{(2C_1 + C_2)^2 + (2D_1 + D_2)^2}$$

(j)

$$u_{20} = \frac{p_0}{6k} \sqrt{(4C_1 - C_2)^2 + (4D_1 - D_2)^2}$$

(k)

These $u_{10}$ can be computed when the amplitude $p_0$ and frequency $\omega$ of the exciting force are known together with system properties $k$, $\omega_n$, and $\zeta_n$.

It can be shown that Eqs. (h) and (i), specialized for $\zeta_n = 0$, are identical to the results for the system without damping obtained in Example 12.2.
Example 12.6

The dynamic response of the system of Fig. E12.6a to the excitation shown in Fig. E12.6b is desired. Determine (a) displacements $u_1(t)$ and $u_2(t)$; (b) bending moments and shears at sections $a$, $b$, $c$, and $d$ as functions of time; (c) the shearing force and bending moment diagrams at $t = 0.18$ sec. The system and excitation parameters are $E = 29,000$ ksi, $I = 100$ in$^4$, $L = 120$ in., $mL = 0.1672$ kip-sec$^2$/in., and $p_o = 5$ kips. Neglect damping.

Solution The mass and stiffness matrices are available from Example 9.5. The natural frequencies and modes of this system were determined in Example 10.2. They are $\omega_1 = 3.156\sqrt{EI/mL^4}$ and $\omega_2 = 16.258\sqrt{EI/mL^4}$; $\phi_1 = (1 \ 0.3274)^T$ and $\phi_2 = (1 \ -1.5274)^T$. Substituting for $E$, $I$, $m$, and $L$ gives $\omega_1 = 10.00$ and $\omega_2 = 51.51$ rad/sec.

![Figure E12.6](image_url)
1. Set up the modal equations.

\[ M_1 = \phi_1^T m \phi_1 = 0.0507 \quad M_2 = \phi_2^T m \phi_2 = 0.2368 \text{ kip-sec}^2/\text{in.} \]

\[ P_1(t) = \phi_1^T \begin{bmatrix} P_0 \\ o \end{bmatrix} = 5 \quad P_2(t) = \phi_2^T \begin{bmatrix} P_0 \\ o \end{bmatrix} = 5 \text{ kips} \]

The modal equations (12.4.6) are

\[ \ddot{q}_1 + 10^2 q_1 = \frac{5}{0.0507} = 98.62 \quad \ddot{q}_2 + (51.51)^2 q_2 = \frac{5}{0.2368} = 21.12 \]  

(a)

2. Solve the modal equations. Adapting the SDF system result, Eq. (4.3.2), to Eq. (a) gives

\[ q_1(t) = \frac{98.62}{10^2} (1 - \cos 10t) = 0.986(1 - \cos 10t) \]  

\[ q_2(t) = \frac{21.12}{(51.51)^2} (1 - \cos 51.51t) = 0.008(1 - \cos 51.51t) \]  

(b)

3. Determine the displacement response. Substituting for \( \phi_1, \phi_2, q_1(t), \) and \( q_2(t) \) in Eq. (12.5.2) gives

\[ u_1(t) = 0.994 - 0.986 \cos 10t - 0.008 \cos 51.51t \]  

\[ u_2(t) = 0.311 - 0.323 \cos 10t + 0.012 \cos 51.51t \]  

(c)

4. Determine the equivalent static forces. Substituting for \( \omega_1^2, m, \) and \( \phi_1 \) in Eq. (12.6.2) gives the forces shown in Fig. E12.6e:

\[ f_1(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}_1 = 10^2 \begin{bmatrix} 0.0418 \\ 0.0836 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3274 \end{bmatrix} q_1(t) = \begin{bmatrix} 4.180 \\ 2.737 \end{bmatrix} q_1(t) \]  

(d)

Similarly substituting \( \omega_2^2, m, \) and \( \phi_2 \) gives the forces shown in Fig. E12.6d:

\[ f_2(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}_2 = \begin{bmatrix} -110.9 \\ -338.8 \end{bmatrix} q_2(t) \]  

(e)

The combined forces are

\[ f_1(t) = 4.180 q_1(t) + 110.9 q_2(t) \quad f_2(t) = 2.737 q_1(t) - 338.8 q_2(t) \]  

(f)

5. Determine the internal forces. Static analysis of the cantilever beam of Fig. E12.6e gives the shearing forces and bending moments at the various sections \( a, b, c, \) and \( d \):

\[ V_a(t) = V_b(t) = f_1(t) \quad V_c(t) = V_d(t) = f_1(t) + f_2(t) \]  

\[ M_a(t) = 0 \quad M_b(t) = \frac{L}{2} f_1(t) \quad M_d(t) = L f_1(t) + \frac{L}{2} f_2(t) \]  

(g)

(h)

where \( f_1(t) \) and \( f_2(t) \) are known from Eqs. (f) and (b).

6. Determine the internal forces at \( t = 0.18 \text{ sec} \). At \( t = 0.18 \text{ sec} \), from Eq. (b), \( q_1 = 1.217 \text{ in.} \) and \( q_2 = 0.0159 \text{ in.} \). Substituting these in Eqs. (d) and (e) gives numerical values for the equivalent static forces shown in Fig. E12.6c and d, wherein the shearing forces and bending moments due to each mode are plotted. The combined values of these element forces are shown in Fig. E12.6e.
Modal contribution of excitation vector $p(t) = sp(t)$

Suppose the applied force vector $p(t)$ has fixed spatial distribution $s$ (pattern vector which is independent of time), but the magnitude of $p(t)$ varies with time as a scalar value $p(t)$.

$$p(t) = sp(t)$$

We can expand the vector $s$ using combination of modes

$$s = \sum_{r=1}^{N} s_r = \sum_{r=1}^{N} \Gamma_r m\phi_r$$

If we pre-multiply by $\phi^T_n$ and use mode orthogonality, we obtain

$$\phi^T_n s = \Gamma_n \phi^T_n m\phi_n = \Gamma_n M_n \phi_n$$

The contribution of nth mode to the excitation vector $s$ is

$$s_n = \Gamma_n m\phi_n$$

The generalized force is

$$P_n(t) = \phi^T_n sp(t) = \sum_{r=1}^{N} \Gamma_r \left( \phi^T_n m\phi_r \right) p(t) = \Gamma_n M_n p(t)$$
Example

\[
m = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ && 1 \end{bmatrix}
\]

\[
k = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix}
\]

\[
\phi_1 = \begin{bmatrix} 0.334 \\ 0.641 \\ 0.895 \\ 1.078 \\ 1.173 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} -0.895 \\ -1.173 \\ -0.641 \\ 0.334 \\ 1.078 \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} 1.173 \\ 0.334 \\ -0.641 \\ -1.078 \\ 0.895 \end{bmatrix}, \quad \phi_4 = \begin{bmatrix} -1.078 \\ 0.895 \\ -1.173 \\ 0.334 \\ 0.641 \end{bmatrix}, \quad \phi_5 = \begin{bmatrix} 0.641 \\ -1.078 \\ -0.895 \\ -0.334 \\ 1.173 \end{bmatrix}
\]

Figure 12.8.1 Uniform five-story shear building.

Figure 12.8.2 Natural modes of vibration of uniform five-story shear building.
Consider two sets of forces

\[
p(t) = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
p(t) \quad \text{and} \quad p(t) = \begin{bmatrix}
0 \\
0 \\
-1 \\
2
\end{bmatrix}
\]

The first one \([0 \ 0 \ 0 \ 0 \ 1]^T\) can be expanded as

\[
s_1 = \begin{bmatrix}
0.101 \\
0.195 \\
0.272 \\
0.327 \\
0.356
\end{bmatrix}
\quad s_2 = \begin{bmatrix}
-0.250 \\
-0.327 \\
-0.179 \\
0.093 \\
0.301
\end{bmatrix}
\quad s_3 = \begin{bmatrix}
0.272 \\
0.077 \\
-0.250 \\
-0.149 \\
0.208
\end{bmatrix}
\quad s_4 = \begin{bmatrix}
-0.179 \\
0.149 \\
0.055 \\
-0.195 \\
0.106
\end{bmatrix}
\quad s_5 = \begin{bmatrix}
0.055 \\
-0.093 \\
0.101 \\
-0.077 \\
0.029
\end{bmatrix}
\]

The second one \([0 \ 0 \ 0 \ -1 \ 2]^T\) can be expanded as

\[
s_1 = \begin{bmatrix}
0.110 \\
0.210 \\
0.294 \\
0.354 \\
0.385
\end{bmatrix}
\quad s_2 = \begin{bmatrix}
-0.423 \\
-0.553 \\
-0.302 \\
0.157 \\
0.508
\end{bmatrix}
\quad s_3 = \begin{bmatrix}
0.739 \\
0.210 \\
-0.679 \\
-0.403 \\
0.564
\end{bmatrix}
\quad s_4 = \begin{bmatrix}
-0.685 \\
0.569 \\
0.212 \\
-0.746 \\
0.407
\end{bmatrix}
\quad s_5 = \begin{bmatrix}
0.259 \\
-0.436 \\
0.475 \\
-0.363 \\
0.135
\end{bmatrix}
\]

Figure 12.8.3  Modal expansion of excitation vectors \(s_p\) and \(s_b\).
Modal analysis for $p(t) = sp(t)$

From the modal equation

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n} = \Gamma_n p(t)$$

The factor $\Gamma_n$ is called modal participation factor. It is a measure of the degree to which the $n$th mode participates in the response.

This modal participation factor is not a useful definition because it depends on how the modes are normalized.

We can write the modal coordinate $q_n(t)$ in term of response of SDF system with a unit mass, and vibration properties of the $n$th mode (natural frequency and damping ratio of the $n$th mode).

$$\ddot{D}_n + 2\zeta_n \omega_n \dot{D}_n + \omega_n^2 D_n = p(t)$$

where

$$q_n(t) = \Gamma_n D_n(t)$$

This form is convenient because we can construct a response spectrum for excitation $p(t)$ and the response of SDF system $D_n(t)$ could be directly read from the spectrum.

The contribution to displacement of the $n$th mode is

$$u_n(t) = \phi_n q_n(t) = \Gamma_n \phi_n D_n(t)$$

The equivalent static force on the $n$th mode is

$$f_n(t) = ku_n(t) = \omega_n^2 m \phi_n q_n(t) = \omega_n^2 m \phi_n \Gamma_n D_n(t) = s_n \left[ \omega_n^2 D_n(t) \right] = s_n A_n(t)$$
The contribution of nth mode to response $r(t)$ is

$$r_n(t) = r_{n}^{st} \left[ \frac{\omega_n^2}{\omega_n^2} D_n(t) \right]$$

where $r_{n}^{st}$ is the static response due to external forces $s_n$

$r_n(t)$ is a product of results from two analyses:

1. Static analysis of the structure subjected to external forces $s_n$
2. Dynamic analysis of the nth mode SDF system subjected to excitation $p(t)$

The combined response due to contribution from all modes is

$$r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_{n}^{st} \left[ \frac{\omega_n^2}{\omega_n^2} D_n(t) \right]$$

**Modal contribution to response**

The contribution to response from the nth mode is

$$r_n(t) = r_{n}^{st} \bar{r}_n \left[ \frac{\omega_n^2}{\omega_n^2} D_n(t) \right]$$

where $r_{n}^{st}$ is the static value of $r$ due to external forces $s$ and

$$\bar{r}_n = \frac{r_{n}^{st}}{r_{n}^{st}}$$

is the nth modal contribution factor. $\bar{r}_n$ is dimensionless and independent of how the modes are normalized. And

$$\sum_{n=1}^{N} \bar{r}_n = 1$$
CHAPTER 13
EARTHQUAKE ANALYSIS OF LINEAR SYSTEMS

RESPONSE HISTORY ANALYSIS

Equation of motion

\[ m \ddot{u} + c \dot{u} + ku = p_{eff}(t) \]

where

\[ p_{eff}(t) = -m \ddot{u}_g(t) \]

The spatial distribution of \( p_{eff}(t) \) is \( s = mu \)

Modal expansion of displacement and forces

\[ u(t) = \sum_{r=1}^{N} \phi_r(t) = \Phi q(t) \]

\[ mu = \sum_{n=1}^{N} \Gamma_n m \phi_n \]

where

\[ \Gamma_n = \frac{L_n}{M_n}, \quad L_n = \phi_n^T m \phi_n, \quad M_n = \phi_n^T m \phi_n \]

The contribution of the nth mode to excitation vector \( mu \) is

\[ s_n = \Gamma_n m \phi_n \]

which is independent of how modes are normalized.

Modal equations

\[ \ddot{q}_n + 2 \zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = -\Gamma_n \ddot{u}_g(t) \]

Rewrite \( q_n(t) \) as \( q_n(t) = \Gamma_n D_n(t) \)

Then,

\[ \ddot{D}_n + 2 \zeta_n \omega_n \dot{D}_n + \omega_n^2 D_n = -\ddot{u}_g(t) \]
Modal responses

Displacement due to the nth mode is
\[ u_n(t) = \phi_n q_n(t) = \Gamma_n \phi_n D_n(t) \]

The equivalent static force in the nth mode is
\[ f_n(t) = s_n A_n(t) \]

where \( A_n(t) = \omega_n^2 D_n(t) \)

The response contribution from the nth mode is
\[ r_n(t) = r_n^{st} A_n(t) \]
\[ u_n(t) = \frac{\Gamma_n \phi_n}{\omega_n^2} A_n(t) \]

Total responses

Total displacement is
\[ u(t) = \sum_{n=1}^{N} u_n(t) = \sum_{n=1}^{N} \Gamma_n \phi_n D_n(t) \]

Total response is
\[ r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{st} A_n(t) \]
<table>
<thead>
<tr>
<th>Mode</th>
<th>Static Analysis of Structure</th>
<th>Dynamic Analysis of SDF System</th>
<th>Modal Contribution to Dynamic Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Forces $s_1$</td>
<td>$A_1(t)$</td>
<td>$r_1(t) = r_1^{s1} A_1(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_1$, $\zeta_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\ddot{u}_d(t)$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Forces $s_2$</td>
<td>$A_2(t)$</td>
<td>$r_2(t) = r_2^{s2} A_2(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_2$, $\zeta_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\ddot{u}_d(t)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N$</td>
<td>Forces $s_N$</td>
<td>$A_N(t)$</td>
<td>$r_N(t) = r_N^{sN} A_N(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\omega_N$, $\zeta_N$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\ddot{u}_d(t)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Total response $r(t) = \sum_{n=1}^{N} r_n(t)$

Figure 13.1.1 Conceptual explanation of modal analysis.
Example 13.1

Determine the response of the inverted L-shaped frame of Fig. E9.6a to horizontal ground motion.

**Solution** Assuming the two elements to be axially rigid, the DOFs are \( u_1 \) and \( u_2 \) (Fig. E9.6a). The equations of motion are given by Eqs. (13.1.1) and (13.1.2), where the influence vector \( \mathbf{1} = (1 \ 0)^T \) (Fig. 9.4.4) and the mass and stiffness matrices (from Example 9.6) are

\[
\mathbf{m} = \begin{bmatrix} 3m \\ m \end{bmatrix}, \quad \mathbf{k} = \frac{6EI}{L^3} \begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}
\]  

(a)

The effective earthquake forces are

\[
\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\ddot{u}_e(t) = -\begin{bmatrix} 3m \\ m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ddot{u}_e(t) = -\begin{bmatrix} 3m \\ 0 \end{bmatrix} \ddot{u}_e(t)
\]  

(b)

The force in the vertical DOF is zero because the ground motion is horizontal.

The natural frequencies and modes of the system are (from Example 10.3)

\[
\omega_1 = 0.6987 \sqrt{\frac{EI}{mL^3}}, \quad \omega_2 = 1.874 \sqrt{\frac{EI}{mL^3}}
\]  

(c)

\[
\Phi_1 = \begin{bmatrix} 1 \\ 2.097 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 1 \\ -1.431 \end{bmatrix}
\]

(d)

Substituting for \( \mathbf{m} \) and \( \mathbf{1} \) in Eq. (13.1.5) gives the first-mode quantities:

\[
L_1 = \Phi_1^T \mathbf{m} \mathbf{1} = (1 \ 2.097) \begin{bmatrix} 3m \\ m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3m
\]

\[
M_1 = \Phi_1^T \mathbf{m} \Phi_1 = (1 \ 2.097) \begin{bmatrix} 3m \\ m \end{bmatrix} \begin{bmatrix} 1 \\ 2.097 \end{bmatrix} = 7.397m
\]

\[
\Gamma_1 = \frac{L_1}{M_1} = \frac{3m}{7.397m} = 0.406
\]

Similar calculations for the second mode give \( L_2 = 3m, \ M_2 = 5.048m, \) and \( \Gamma_2 = 0.594. \)

Substituting \( \Gamma_n, \mathbf{m}, \) and \( \Phi_n \) in Eq. (13.1.6) gives

\[
\mathbf{s}_1 = \Gamma_1 \mathbf{m} \Phi_1 = 0.406 \begin{bmatrix} 3m \\ m \end{bmatrix} \begin{bmatrix} 1 \\ 2.097 \end{bmatrix} = m \begin{bmatrix} 1.218 \\ 0.851 \end{bmatrix}
\]

(e)

\[
\mathbf{s}_2 = \Gamma_2 \mathbf{m} \Phi_2 = 0.594 \begin{bmatrix} 3m \\ m \end{bmatrix} \begin{bmatrix} 1 \\ -1.431 \end{bmatrix} = m \begin{bmatrix} 1.782 \\ -0.851 \end{bmatrix}
\]

(f)

Then Eq. (13.1.4) specializes to

\[
m \begin{bmatrix} 3 \\ 0 \end{bmatrix} = m \begin{bmatrix} 1.218 \\ 0.851 \end{bmatrix} + m \begin{bmatrix} 1.782 \\ -0.851 \end{bmatrix}
\]

(g)

This modal expansion of the spatial distribution of effective forces is shown in Fig. E13.1. Observe that the forces along the vertical DOF in the two modes cancel each other because the effective earthquake force in this DOF is zero.

Substituting for \( \Gamma_n \) and \( \Phi_n \) in Eq. (13.1.10) gives the first-mode displacements

\[
\mathbf{u}_1(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \Gamma_1 \Phi_1 \mathbf{D}_1(t) = 0.406 \begin{bmatrix} 1 \\ 2.097 \end{bmatrix} \begin{bmatrix} 0.406 \\ 0.851 \end{bmatrix} \mathbf{D}_1(t)
\]

(h)
and the second-mode displacements

\[
\mathbf{u}_2(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_2 = \Gamma_2 \phi_2 D_2(t) = 0.594 \begin{bmatrix} 1 \\ -1.431 \end{bmatrix} D_2(t) = \begin{bmatrix} 0.594 \\ -0.851 \end{bmatrix} D_2(t)
\]

(i)

Combining Eqs. (h) and (i) gives the total displacements:

\[
u_1(t) = 0.406 D_1(t) + 0.594 D_2(t) \quad u_2(t) = 0.851 D_1(t) - 0.851 D_2(t)
\]

(j)

The earthquake-induced bending moment \( M_b \) at the base of the column due to the \( n \)th mode [from Eq. (13.1.13)] is

\[
M_{bn}(t) = M_{bn}^{st} A_n(t)
\]

(k)

Static analyses of the frame for the forces \( s_1 \) and \( s_2 \) give \( M_{b1}^{st} \) and \( M_{b2}^{st} \) as shown in Fig. E13.1. Substituting for \( M_{bn}^{st} \) and combining modal contributions gives the total bending moment:

\[
M_b(t) = \sum_{n=1}^{2} M_{bn}(t) = 2.069mL A_1(t) + 0.931mL A_2(t)
\]

(l)

The three response quantities considered have been, and other responses can be, expressed in terms of \( D_n(t) \) and \( A_n(t) \). These responses of the \( n \)th-mode SDF system to given ground acceleration \( \ddot{u}_g(t) \) can be determined by numerical time-stepping methods (Chapter 5).

---

**Figure E13.1**
Multistory buildings with symmetric plan

\[ m\ddot{u} + c\dot{u} + ku = -m1\ddot{u}_g(t) \]

**Figure 13.2.1** Dynamic degrees of freedom of a multistory frame: lateral displacements relative to the ground.

\[ m1 = \sum_{n=1}^{N} s_n = \sum_{n=1}^{N} \Gamma_n m\phi_n \]

where

\[ \Gamma_n = \frac{L^h_n}{M_n} \]

\[ L^h_n = \sum_{j=1}^{N} m_j \phi_{jn} \]

\[ M_n = \phi_n^T m\phi_n = \sum_{j=1}^{N} m_j \phi_{jn}^2 \]

\( m_j \) is the mass of the jth story.

**TABLE 13.2.1 MODAL STATIC RESPONSES**

<table>
<thead>
<tr>
<th>Response, ( r )</th>
<th>Modal Static Response, ( r_{n}^{st} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_i )</td>
<td>( V_{in}^{st} = \sum_{j=1}^{N} s_{jn} )</td>
</tr>
<tr>
<td>( M_i )</td>
<td>( M_{in}^{st} = \sum_{j=1}^{N} (h_j - h_i) s_{jn} )</td>
</tr>
<tr>
<td>( V_b )</td>
<td>( V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n L^h_n \equiv M_n^* )</td>
</tr>
<tr>
<td>( M_b )</td>
<td>( M_{bn}^{st} = \sum_{j=1}^{N} h_j s_{jn} = \Gamma_n L^\theta_n \equiv h_n^{<em>} M_n^</em> )</td>
</tr>
<tr>
<td>( u_j )</td>
<td>( u_{jn}^{st} = (\Gamma_n/\omega_n^2) \phi_{jn} )</td>
</tr>
<tr>
<td>( \Delta_j )</td>
<td>( \Delta_{jn}^{st} = (\Gamma_n/\omega_n^2)(\phi_{jn} - \phi_{j-1,n}) )</td>
</tr>
</tbody>
</table>
Example 13.2

A two-story shear frame has the mass and story stiffnesses properties shown in Fig. E13.2a. Determine the modal expansion of the effective earthquake force distribution associated with horizontal ground acceleration \( \ddot{u}_g(t) \).

**Solution** The stiffness and mass matrices (from Example 9.1) are

\[
\mathbf{k} = k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{m} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}
\]

where \( k = 24EI_c/h^3 \), and the natural frequencies and modes (from Example 10.4) are

\[
\omega_1 = \sqrt{\frac{k}{2m}} \quad \omega_2 = \sqrt{\frac{2k}{m}}
\]

\[
\phi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

The modal properties \( M_n, L_n^h, \) and \( \Gamma_n \) are computed from Eq. (13.2.3). For the first mode:

\( M_1 = 2m \left( \frac{1}{2} \right)^2 + m(1)^2 = 3m/2 \); \( L_1^h = 2m \left( \frac{1}{2} \right) + m(1) = 2m \); \( \Gamma_1 = L_1^h/M_1 = \frac{4}{3} \). Similarly, for the second mode: \( M_2 = 3m \), \( L_2^h = -m \), and \( \Gamma_2 = \frac{1}{3} \). Substituting for \( \Gamma_n, \mathbf{m}, \) and \( \phi_n \) in Eq. (13.2.4) gives

\[
s_1 = \frac{4}{3} m \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \frac{4}{3} m \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
s_2 = -\frac{1}{3} m \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{3} m \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

The modal expansion of \( \mathbf{u}_1 \) is displayed in Fig. E13.2b.

![Figure E13.2](image)

**Figure E13.2** (a) Two-story shear frame; (b) modal expansion of \( \mathbf{u}_1 \).
Example 13.3

Derive equations for (a) the floor displacements and (b) the story shears for the shear frame of Example 13.2 subjected to ground motion \( \ddot{u}_g(t) \).

**Solution** Steps 1 to 4 of the procedure summary have already been implemented in Example 13.2.

(a) *Floor displacements.* Substituting \( \Gamma_n \) and \( \phi_{jn} \) from Example 13.2 in Eq. (13.2.5) gives the floor displacements due to the each mode:

\[
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_1 = \frac{4}{3} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} D_1(t) \quad \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}_2 = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} D_2(t)
\]

(a)

Combining the contributions of the two modes gives the floor displacements:

\[
\begin{align*}
    u_1(t) &= u_{11}(t) + u_{12}(t) = \frac{2}{3} D_1(t) + \frac{1}{3} D_2(t) \\
    u_2(t) &= u_{21}(t) + u_{22}(t) = \frac{4}{3} D_1(t) - \frac{1}{3} D_2(t)
\end{align*}
\]

(b)

(b) *Story shears.* Static analysis of the frame for external floor forces \( s_i \) gives \( V_{in}^{st} \), \( i = 1 \) and 2, shown in Fig. E13.3. Substituting these results in Eq. (13.2.8) gives

\[
\begin{align*}
    V_{11}(t) &= \frac{8}{3} m A_1(t) \\
    V_{21}(t) &= \frac{4}{3} m A_1(t) \\
    V_{12}(t) &= \frac{1}{3} m A_2(t) \\
    V_{22}(t) &= -\frac{1}{3} m A_2(t)
\end{align*}
\]

Effective modal mass and heights

\[
V_{bn}^*(t) = V_{bn}^{st} A_n(t) = M_n^* A_n(t)
\]

where \( M_n^* = V_{bn}^{st} = \Gamma_n L_n^h \frac{(L_n^h)^2}{M_n} \) is called the effective modal mass.

\[
\sum_{n=1}^{N} M_n^* = \sum_{n=1}^{N} m_j
\]

\[
M_{bn}^*(t) = M_n^{st} A_n(t) = h_n^* V_{bn}(t)
\]

\[
h_n^* = \frac{L_n^\theta}{L_n^h} \text{ is called the effective modal height where } L_n^\theta = \sum_{j=1}^{N} h_j m_j \phi_{jn}
\]

\[
\sum_{n=1}^{N} h_n^* M_n^* = \sum_{n=1}^{N} h_j m_j
\]
Example

Floor Mass  Story Stiffness

\[ u_5 \quad m \quad k \]
\[ u_4 \quad m \quad k \]
\[ u_3 \quad m \quad k \]
\[ u_2 \quad m \quad k \]
\[ u_1 \quad m \quad k \]

Figure 12.8.1 Uniform five-story shear building.

\[ m \quad 1.252m \quad 0.362m \quad 0.159m \quad -0.063m \quad 0.015m \]
\[ m \quad 1.150m \quad -0.112m \quad -0.113m \quad 0.116m \quad -0.040m \]
\[ m \quad 0.956m \quad 0.215m \quad -0.191m \quad -0.033m \quad 0.053m \]
\[ m \quad 0.684m \quad 0.394m \quad 0.059m \quad -0.088m \quad -0.049m \]
\[ m \quad 0.356m \quad 0.301m \quad 0.208m \quad 0.106m \quad 0.029m \]

Figure 13.2.4 Modal expansion of \( m_1 \).

### Table 13.2.2 Modal Properties

<table>
<thead>
<tr>
<th>Mode</th>
<th>( M_n )</th>
<th>( L_n^h )</th>
<th>( L_n^\theta / h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.067</td>
<td>3.750</td>
</tr>
<tr>
<td>2</td>
<td>1.000</td>
<td>-0.336</td>
<td>0.404</td>
</tr>
<tr>
<td>3</td>
<td>1.000</td>
<td>0.177</td>
<td>0.135</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
<td>-0.099</td>
<td>0.059</td>
</tr>
<tr>
<td>5</td>
<td>1.000</td>
<td>0.045</td>
<td>0.023</td>
</tr>
</tbody>
</table>
TABLE 13.2.3 MODAL STATIC RESPONSES

<table>
<thead>
<tr>
<th>Mode</th>
<th>$V_{bn}^{st}/m$</th>
<th>$V_{5n}^{st}/m$</th>
<th>$M_{bn}^{st}/mh$</th>
<th>$u_{5n}^{st}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.398</td>
<td>1.252</td>
<td>15.45</td>
<td>0.127</td>
</tr>
<tr>
<td>2</td>
<td>0.436</td>
<td>-0.362</td>
<td>-0.525</td>
<td>-0.004</td>
</tr>
<tr>
<td>3</td>
<td>0.121</td>
<td>0.159</td>
<td>0.092</td>
<td>0.0008</td>
</tr>
<tr>
<td>4</td>
<td>0.037</td>
<td>-0.063</td>
<td>-0.022</td>
<td>-0.0002</td>
</tr>
<tr>
<td>5</td>
<td>0.008</td>
<td>0.015</td>
<td>0.004</td>
<td>0.00003</td>
</tr>
</tbody>
</table>

Figure 13.2.5 Effective modal masses and effective modal heights.

\[ u_n(t) = \Gamma_n \phi_n D_n(t) \]

\[ u_{rn}(t) = \Gamma_n \phi_{rn} D_n(t) \]

\[ V_{bn}(t) = V_{bn}^{st} A_n(t) = M_n^{st} A_n(t) \]

\[ M_{bn}(t) = M_{bn}^{st} A_n(t) = h_n^{st} V_{bn}(t) \]
Figure 13.2.6 Displacement $D_n(t)$ and pseudo-acceleration $A_n(t)$ responses of modal SDF systems.

Figure 13.2.7 Base shear and fifth-story shear: modal contributions, $V_{nm}(t)$ and $V_{nl}(t)$, and total responses, $V_b(t)$ and $V_5(t)$.
Figure 13.2.8 Roof displacement and base overturning moment: modal contributions, $u_5(t)$ and $M_{im}(t)$, and total responses, $u_5(t)$ and $M_b(t)$. 
Response Spectrum Analysis (RSA)

To determine the peak value of response due to earthquake excitation, the modal response history analysis requires computation of response at all time instants during vibration, i.e, \( D_n(t) \) and \( A_n(t) \) by solving the equations of motion for modal SDF systems,

\[
\ddot{D}_n + 2\zeta_n\omega_n\dot{D}_n + \omega_n^2 D_n = p(t)
\]

\[
A_n(t) = D_n(t)\omega_n^2
\]

and \( r_n(t) \) and \( r(t) \), from the equations.

Modal response

\[
r_n(t) = r_{nt}^A_n(t)
\]

Total response

\[
r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_{nt}^A_n(t)
\]

Finally, the peak response

\[
r_o = \max_{\forall t} |r(t)|
\]

To avoid solving the equation of motions for modal SDF systems, the peak value of \( D_n(t) \) and \( A_n(t) \) can be conveniently read from response spectrum, so the peak modal response can be calculated as

Peak modal response

\[
r_{no} = r_{nt}^A_n
\]

where

\[
A_n = \max_{\forall t} |A_n(t)|
\]

The peak response could be estimated by combining the peak modal response using modal combination rule, such as square-root-of-sum-of-squares (SRSS) or complete quadratic combination (CQC) rule.
SRSS \[ r_o \approx \sqrt{\sum_{n=1}^{N} r_{no}^2} \]

CQC \[ r_o \approx \sqrt{\sum_{i=1}^{N} \sum_{n=1}^{N} \rho_{in}^2 r_i r_{no}} \]

where \[ \rho_{in} = \frac{8 \sqrt{\zeta_i \zeta_n (\zeta_i + \beta_{in} \zeta_n)} \beta_{in}^{3/2}}{(1 - \beta_{in}^2)^2 + 4 \zeta_i \zeta_n \beta_{in} (1 + \beta_{in}^2) + 4 \left( \zeta_i^2 + \zeta_n^2 \right) \beta_{in}^2} \]

Response history analysis (RHA): peak base shear = 73.728 kips.

Response spectrum analysis (RSA): peak base shear is

\[
\text{SRSS} \quad \sqrt{60.469^2 + 24.533^2 + 9.867^2 + 2.943^2 + 0.595^2} = 66.066 \text{ kips}
\]

The difference is because the peaks of different modes do not occur at the same time, so the peak of total response cannot be computed exactly.
Rayleigh Damping Matrix (section 11.4.1)

Damping matrix should not be computed from dimension structural member. Instead, it should be constructed from modal damping ratios.

To construct a classical damping matrix, which give a diagonal matrix $C = \Phi^T c \Phi$, Rayleigh damping matrix $c = a_0 m + a_1 k$ uses combination of mass-proportional damping matrix and stiffness-proportional damping matrix. Each term retains the orthogonal properties of mode shapes.

For mass-proportional damping $C = \Phi^T c \Phi = a_0 \Phi^T m \Phi = a_o M$

having diagonal entries $C_n = a_0 M_n = 2 \zeta_n \omega_n M_n$

$$a_0 = 2 \omega_n \zeta_n \quad \text{or} \quad \zeta_n = \frac{a_0}{2 \omega_n}$$

For stiffness-proportional damping $C = \Phi^T c \Phi = a_1 \Phi^T k \Phi = a_i K$

having diagonal entries $C_n = a_i K_n = a_i \omega_n^2 M_n$

$$a_i \omega_n^2 = 2 \omega_n \zeta_n \quad \text{or} \quad \zeta_n = \frac{a_i \omega_n}{2}$$

Thus for $c = a_0 m + a_1 k \quad \Rightarrow \quad \zeta_n = \frac{a_0}{2 \omega_n} + \frac{a_i \omega_n}{2}$

The coefficient $a_0$ and $a_i$ depends on the modal damping ratios selected for two modes. Suppose we want the damping ratios of mode $i$ and $j$ to be $\zeta_i$ and $\zeta_j$, and the modal frequencies are $\omega_i$ and $\omega_j$. The coefficients can be determined by solving

$$\begin{bmatrix} 1/\omega_i & \omega_i \\ 1/\omega_j & \omega_j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \zeta_i \\ \zeta_j \end{bmatrix}$$
Example 11.1

The properties of a three-story shear building are given in Fig. E11.1. These include the floor weights, story stiffnesses, natural frequencies, and modes. Derive a Rayleigh damping matrix such that the damping ratio is 5% for the first and second modes. Compute the damping ratio for the third mode.

\[
\begin{align*}
\text{Figure E11.1}
\end{align*}
\]

Solution

1. Set up the mass and stiffness matrices.

\[
\mathbf{m} = \frac{1}{386} \begin{bmatrix}
400 & 400 \\
400 & 200 \\
\end{bmatrix}, \quad \mathbf{k} = 610 \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1 \\
\end{bmatrix}
\]

2. Determine \(a_0\) and \(a_1\) from Eq. (11.4.9).

\[
\begin{bmatrix}
1/12.57 & 12.57 \\
1/34.33 & 34.33 \\
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
\end{bmatrix} = 2 \begin{bmatrix}
0.05 \\
0.05 \\
\end{bmatrix}
\]

These algebraic equations have the following solution:

\(a_0 = 0.9198, \quad a_1 = 0.0021\)

3. Evaluate the damping matrix.

\[
\mathbf{c} = a_0 \mathbf{m} + a_1 \mathbf{k} = \begin{bmatrix}
3.55 & -1.30 & 0 \\
1.78 & 3.55 & -1.30 \\
\end{bmatrix}
\]

4. Compute \(\zeta_3\) from Eq. (11.4.8).

\[
\zeta_3 = \frac{0.9198}{2(46.89)} + \frac{0.0021(46.89)}{2} = 0.0593
\]
If modal damping ratios are to be specified for all modes and the classical damping matrix is to be obtained, the generalized damping matrix is

\[
\begin{bmatrix}
C_1 \\
C_2 \\
.. \\
C_n
\end{bmatrix} =
\begin{bmatrix}
2\zeta_1 M_1 \omega_1 \\
2\zeta_2 M_2 \omega_2 \\
.. \\
2\zeta_n M_n \omega_n
\end{bmatrix}
\]

And we know that

\[
C = \Phi^T c \Phi
\]

Therefore, we multiply \((\Phi^T)^{-1}\) from the left and \(\Phi^{-1}\) from the right; we get

\[
c = (\Phi^T)^{-1} C \Phi^{-1}
\]

This is the classical damping matrix which results in modal damping ratios that we specified.
Example

\[ \m = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \k = \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ \omega_n = 0.445, 1.247, 1.802 \]

\[ \Phi = \begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix} \]

\[ \M = \Phi^T \m \Phi = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \C = \begin{bmatrix} 2 \zeta_1 M_1 \omega_1 \\ 2 \zeta_2 M_2 \omega_2 \\ 2 \zeta_3 M_3 \omega_3 \end{bmatrix} = \begin{bmatrix} 0.0445 \\ 0.1247 \\ 0.1802 \end{bmatrix} \]

\[ \c = \left( \Phi^T \right)^{-1} \C \Phi^{-1} = \begin{bmatrix} 0.0871 & -0.0483 & -0.0086 \\ -0.0483 & 0.1268 & -0.0397 \\ -0.0086 & -0.0397 & 0.1355 \end{bmatrix} \]