CHAPTER 9
MULTI-DEGREE-OF-FREEDOM SYSTEMS
Equations of Motion, Problem Statement,
and Solution Methods

Two-story shear building
A shear building is the building whose floor systems are rigid
in flexure and several factors are neglected, for example, axial
deformation of beams and columns.

We will formulate the equations of motion of a simple 2-story
shear building whose mass are lumped at the floor.

The equations of motion are formulated by considering
equilibrium of forces acting on each mass. Any of the two
approaches can be used

(1) Newton’s second law of motion
(2) D’Alembert’s principle of dynamic equilibrium
Newton’s second law of motion

\[ \sum F = m\ddot{u} \]

For each floor mass \((j=1 \text{ and } 2)\)

\[ p_j - f_{sj} - f_{Dj} = m_j\ddot{u}_j \quad \text{or} \quad m_j\ddot{u}_j + f_{sj} + f_{Dj} = p_j(t) \]

Two equations can be written in matrix form

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2 \\
\end{bmatrix}
\begin{bmatrix}
    \ddot{u}_1 \\
    \ddot{u}_2 \\
\end{bmatrix}
+
\begin{bmatrix}
    f_{D1} \\
    f_{D2} \\
\end{bmatrix}
+
\begin{bmatrix}
    f_{S1} \\
    f_{S2} \\
\end{bmatrix}
=
\begin{bmatrix}
    p_1(t) \\
    p_2(t) \\
\end{bmatrix}
\]

\[ m\ddot{u} + f_D + f_S = p(t) \]

where

\[
\begin{align*}
    u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
    m &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \\
    f_D &= \begin{bmatrix} f_{D1} \\ f_{D2} \end{bmatrix} \\
    f_S &= \begin{bmatrix} f_{S1} \\ f_{S2} \end{bmatrix} \\
    p &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}
\end{align*}
\]

Because all beams are assumed rigid, the story shear force can be directly related to the relative displacement between stories.

\[ V_j = k_j\Delta_j \]

where \( \Delta_j = u_{j+1} - u_j \) and \( k_j = \sum_{\text{columns}} \frac{12EI_c}{h^3} \)

The elastic force acting on the first story mass comes from columns below \((f_{S1}^b)\) and above \((f_{S1}^a)\) the floor.

\[
\begin{align*}
    f_{S1} &= f_{S1}^b + f_{S1}^a \\
    f_{S1}^b &= k_1u_1 + k_2(u_1 - u_2) \\
    f_{S2} &= k_2(u_2 - u_1)
\end{align*}
\]
The elastic resisting force vector \( \mathbf{f}_s \) is related to displacement vector \( \mathbf{u} \) through the stiffness matrix \( \mathbf{k} \).

The damping forces \( f_{D1} \) and \( f_{D2} \) are related to floor velocities \( \dot{u}_1 \) and \( \dot{u}_2 \). The \( j^{th} \) story damping coefficient \( c_j \) relates the story shear \( V_j \) due to the damping effects to the velocity \( \dot{\Delta}_j \) associated with the story deformation by

\[
V_j = c_j \dot{\Delta}_j
\]

We can derive

\[
f_{D1} = c_1 \dot{u}_1 + c_2 (\dot{u}_1 - \dot{u}_2)
\]

\[
f_{D2} = c_2 (\dot{u}_2 - \dot{u}_1)
\]

\[
\begin{bmatrix}
  f_{D1} \\
  f_{D2}
\end{bmatrix}
= \begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
  \dot{u}_1 \\
  \dot{u}_2
\end{bmatrix}
\quad \text{or} \quad
\mathbf{f}_D = \mathbf{c} \dot{\mathbf{u}}
\]

The damping force vector \( \mathbf{f}_D \) and velocity vector \( \dot{\mathbf{u}} \) are related through the damping matrix \( \mathbf{c} \).

Therefore, the equations of motion are

\[
\mathbf{m} \ddot{\mathbf{u}} + \mathbf{c} \dot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{p}(t)
\]

This matrix equation represents two ordinary differential equations governing the displacements \( u_1(t) \) and \( u_2(t) \) of the two-story frame subjected external dynamic forces \( p_1(t) \) and \( p_2(t) \).

Each equation contains both unknowns \( u_1 \) and \( u_2 \), so two equations are coupled and must be solved simultaneously.
Dynamic Equilibrium (D’Alembert’s principle)

For each of the mass in the system, the external force must be in balance with

1. inertia force (resisting acceleration) acting in the opposite direction to acceleration
2. damping force (resisting velocity) acting in the opposite direction to velocity and
3. elastic force resisting deformation

Figure 9.1.2 Free-body diagrams.
Example 9.1a

Formulate the equations of motion for the two-story shear frame shown in Fig. E9.1a.

Solution  Equation (9.1.11) is specialized for this system to obtain its equation of motion. To do so, we note that

\[ m_1 = 2m \quad m_2 = m \]

\[ k_1 = 2 \frac{12(2EI_c)}{h^3} = \frac{48EI_c}{h^3} \quad k_2 = 2 \frac{12(EI_c)}{h^3} = \frac{24EI_c}{h^3} \]

Substituting these data in Eqs. (9.1.2) and (9.1.7) gives the mass and stiffness matrices:

\[ m = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad k = \frac{24EI_c}{h^3} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \]

Substituting these \( m \) and \( k \) in Eq. (9.1.11) gives the governing equations for this system without damping:

\[ m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + 24 \frac{EI_c}{h^3} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \]

Observe that the stiffness matrix is nondiagonal, implying that the two equations are coupled, and in their present form must be solved simultaneously.

Figure E9.1a
Example 9.1b

Formulate the equations of motion for the two-story shear frame in Fig. E9.1a using influence coefficients.

Solution

The two DOFs of this system are shown in Fig. E9.1a; thus, $\mathbf{u} = (u_1 \quad u_2)^T$.

\[ u_1 = 1, \quad u_2 = 0 \]

\[ k_{21} \quad u_1 = 1 \]

\[ k_{11} \]

\[ k_1 \]

\[ k_2 \]

(a) \hspace{1cm} (b)

\[ u_2 = 1, \quad u_1 = 0 \]

\[ u_2 = 1 \]

\[ k_{22} \]

\[ k_{12} \]

\[ k_1 \]

\[ k_2 \]

(c) \hspace{1cm} (d)

Figure E9.1b

1. Determine the stiffness matrix. To obtain the first column of the stiffness matrix, we impose $u_1 = 1$ and $u_2 = 0$. The stiffness influence coefficients are $k_{ij}$ (Fig. E9.1b). The forces necessary at the top and bottom of each story to maintain the deflected shape are expressed in terms of story stiffnesses $k_1$ and $k_2$ [part (b) of the figure], defined in Section 9.1.1 and determined in Example 9.1a:

\[ k_1 = \frac{48EI_c}{h^3} \quad k_2 = \frac{24EI_c}{h^3} \]  \hspace{1cm} (a)

The two sets of forces in parts (a) and (b) of the figure are one and the same. Thus,

\[ k_{11} = k_1 + k_2 = \frac{72EI_c}{h^3} \quad k_{21} = -k_2 = -\frac{24EI_c}{h^3} \]  \hspace{1cm} (b)
The second column of the stiffness matrix is obtained in a similar manner by imposing \( u_2 = 1 \) with \( u_1 = 0 \). The stiffness influence coefficients are \( k_{i2} \) [part (c) of the figure] and the forces necessary to maintain the deflected shape are shown in part (d) of the figure. The two sets of forces in parts (c) and (d) of the figure are one and the same. Thus,

\[
k_{12} = -k_2 = -\frac{24EI_c}{h^3} \quad k_{22} = k_2 = \frac{24EI_c}{h^3}
\]  

(c)

With the stiffness influence coefficients determined, the stiffness matrix is

\[
\mathbf{k} = \frac{24EI_c}{h^3} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}
\]  

(d)

2. Determine the mass matrix. With the DOFs defined at the locations of the lumped masses, the diagonal mass matrix is given by Eq. (9.2.10):

\[
\mathbf{m} = m \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}
\]  

(e)

3. Determine the equations of motion. The governing equations are

\[
\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)
\]  

(f)

where \( \mathbf{m} \) and \( \mathbf{k} \) are given by Eqs. (e) and (d), and \( \mathbf{p}(t) = (p_1(t) \quad p_2(t))^T \).
General Approach for Linear Systems

Discretization

A frame structure can be idealized by an assemblage of elements—beams, columns, walls—interconnected at nodal points or nodes. Displacements of nodes are degrees of freedom. A node in a planar two-dimension frame has 3 DOFs—two translations and one rotation.

If axial deformations are neglected, the number of DOFs can be reduced because some translational DOF are equal.

The external forces are applied at the nodes which correspond to the DOFs.
**Elastic forces**

The elastic forces are related to displacement through stiffness matrix. The stiffness matrix can be obtained from stiffness influence coefficient $k_{ij}$, which is the force required along DOF $i$ due to a unit displacement at DOF $j$ and zero displacement at all other DOFs.

For example, the force $k_{ii} (i=1,2,...,8)$ are required to maintain the deflected shape associated with $u_i = 1$ and all other $u_j = 0$.

![Diagram of forces and displacements](image)

**Figure 9.2.3** (a) Stiffness component of frame; (b) stiffness influence coefficients for $u_1 = 1$; (c) stiffness influence coefficients for $u_4 = 1$.

The force $f_{Si}$ at DOF $i$ associated with displacement $u_j$ ($j = 1$ to $N$) is obtained by superposition:

$$f_{Si} = k_{i1}u_1 + k_{i2}u_2 + \ldots + k_{iN}u_N$$
Such equation applies to each of $f_{Si}$ where $i = 1$ to $N$, so

$$
\begin{bmatrix}
  f_{S1} \\
  f_{S2} \\
  \vdots \\
  f_{SN}
\end{bmatrix} =
\begin{bmatrix}
  k_{11} & k_{12} & \cdots & k_{1N} \\
  k_{21} & k_{22} & \cdots & k_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  k_{N1} & k_{N2} & \cdots & k_{NN}
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_N
\end{bmatrix}
$$

or $f_s = ku$

where $k$ is the stiffness matrix of the structure.

This approach can be cumbersome for complex structures in order to visualize a deflected shape with a unit displacement at DOF $j$ and zero displacement at all other DOFs.

The direct stiffness method must be used instead. It involves assembling of stiffness matrices of structural members into the stiffness matrix of the whole system. The appropriate method should be used for a given problem.

**Damping forces**

Damping forces are related to velocities of nodes through damping matrix. The method of damping influence coefficient $c_{ij}$ can be used to derive the damping matrix in a similar manner as stiffness matrix relating elastic forces to displacements.

However, it is impractical to compute the coefficient $c_{ij}$ of damping matrix directly from the size of the structural elements. Instead, damping of a MDF system is usually specified in term of damping ratio and the corresponding damping matrix can be constructed accordingly.
**Inertia forces**

Inertia forces are forces related to acceleration of the mass. An approach to consider inertia forces acting at nodes is to lump the mass of structural components to nodes.

Inertial forces are related to acceleration at nodes through the mass matrix \( m \). Mass matrix can be derived using mass influence coefficient \( m_{ij} \) which is the external force in DOF \( i \) due to unit acceleration along DOF \( j \). For example, the force \( m_{i1} (i=1,2,...,8) \) are required in various DOF to equilibrate the inertia forces associated with \( \ddot{u}_i = 1 \) and all other \( \ddot{u}_j = 0 \).

The force at DOF \( i \) due to acceleration at various nodes can be obtained by superposition

\[
f_i = m_{i1}\ddot{u}_1 + m_{i2}\ddot{u}_2 + ... + m_{iN}\ddot{u}_N
\]

Such inertia forces at all DOFs are written together in the inertia force vector \( f_i \), which is equal to

\[
\begin{bmatrix}
  f_{i1} \\
  f_{i2} \\
  \vdots \\
  f_{iN}
\end{bmatrix} = \begin{bmatrix}
  m_{11} & m_{12} & \cdots & m_{1N} \\
  m_{21} & m_{22} & \cdots & m_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{N1} & m_{N2} & \cdots & m_{NN}
\end{bmatrix} \begin{bmatrix}
  \dddot{u}_1 \\
  \dddot{u}_2 \\
  \vdots \\
  \dddot{u}_N
\end{bmatrix}
\]

or

\[
f_i = m\dddot{u}
\]

When lumped-mass model is used, the mass matrix will be diagonal. Rotational inertia forces at the nodes are neglected, so the mass associated with rotational DOFs are zero.

\[
m_{ij} = 0 \text{ if } i \neq j \quad \text{ or } \quad m_{jj} = m_j \quad \text{ or } \quad 0
\]
Equations of motion

\[ f_i + f_D + f_S = p(t) \]
\[ m\ddot{u} + c\dot{u} + k u = p(t) \]

The off-diagonal terms in the coefficient matrices \( m \), \( c \), and \( k \) are known as coupling terms. The coupling in a system also depends on the choice of DOFs.

Example 9.2

A uniform rigid bar of total mass \( m \) is supported on two springs \( k_1 \) and \( k_2 \) at the two ends and subjected to dynamic forces shown in Fig. E9.2a. The bar is constrained so that it can move only vertically in the plane of the paper; with this constraint the system has two DOFs.
Formulate the equations of motion with respect to displacements $u_1$ and $u_2$ of the two ends as the two DOFs.

**Solution**

1. **Determine the applied forces.** The external forces do not act along the DOFs and should therefore be converted to equivalent forces $p_1$ and $p_2$ along the DOFs (Fig. E9.2b) using equilibrium equations. This can also be achieved by the principle of virtual displacements. Thus if we introduce a virtual displacement $\delta u_1$ along DOF 1, the work done by the applied forces is

$$\delta W = p_1 \frac{\delta u_1}{2} - p_0 \frac{\delta u_1}{L} \quad (a)$$

Similarly, the work done by the equivalent forces is

$$\delta W = p_1 \delta u_1 + p_2 (0) \quad (b)$$

Because the work done by the two sets of forces should be the same, we equate Eqs. (a) and (b) and obtain

$$p_1 = \frac{p_1}{2} - \frac{p_0}{L} \quad (c)$$

In a similar manner, by introducing a virtual displacement $\delta u_2$, we obtain

$$p_2 = \frac{p_1}{2} + \frac{p_0}{L} \quad (d)$$

2. **Determine the stiffness matrix.** Apply a unit displacement $u_1 = 1$ with $u_2 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients $k_{11}$ and $k_{21}$ (Fig. E9.2c). By statics, $k_{11} = k_1$ and $k_{21} = 0$. Now apply a unit displacement $u_2 = 1$ with $u_1 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients (Fig. E9.2d). By statics, $k_{12} = 0$ and $k_{22} = k_2$. Thus the stiffness matrix is

$$k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad (e)$$

In this case the stiffness matrix is diagonal (i.e., there are no coupling terms) because the two DOFs are defined at the locations of the springs.

3. **Determine the mass matrix.** Impart a unit acceleration $\ddot{u}_1 = 1$ with $\ddot{u}_2 = 0$, determine the distribution of accelerations of (Fig. E9.2e) and the associated inertia forces, and identify mass influence coefficients (Fig. E9.2f). By statics, $m_{11} = m/3$ and $m_{21} = m/6$. Similarly, imparting a unit acceleration $\ddot{u}_2 = 1$ with $\ddot{u}_1 = 0$, defining the inertia forces and mass influence coefficients, and applying statics gives $m_{12} = m/6$ and $m_{22} = m/3$. Thus the mass matrix is

$$m = \frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (f)$$

The mass matrix is coupled, as indicated by the off-diagonal terms, because the mass is distributed and not lumped at the locations where the DOFs are defined.

4. **Determine the equations of motion.** Substituting Eqs. (c)–(f) in Eq. (9.2.12) with $c = 0$ gives

$$\frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (p_1/2) - (p_0/L) \\ (p_1/2) + (p_0/L) \end{bmatrix} \quad (g)$$

The two differential equations are coupled because of mass coupling due to the off-diagonal terms in the mass matrix.
Example 9.3

Formulate the equations of motion of the system of Fig. E9.2a with the two DOFs defined at the center of mass $O$ of the rigid bar: translation $u_t$ and rotation $u_\theta$ (Fig. E9.3a).

Solution

1. **Determine the stiffness matrix.** Apply a unit displacement $u_t = 1$ with $u_\theta = 0$ and identify the resulting elastic forces and $k_{tt}$ and $k_{t\theta}$ (Fig. E9.3b). By statics, $k_{tt} = k_1 + k_2$ and $k_{t\theta} = (k_2 - k_1)L/2$. Now, apply a unit rotation $u_\theta = 1$ with $u_t = 0$ and identify the resulting elastic forces and $k_{t\theta}$ and $k_{\theta\theta}$ (Fig. E9.3c). By statics, $k_{t\theta} = (k_2 - k_1)L/2$ and $k_{\theta\theta} = (k_1 + k_2)L^2/4$. Thus the stiffness matrix is

$$\bar{k} = \begin{bmatrix} k_1 + k_2 & (k_2 - k_1)L/2 \\ (k_2 - k_1)L/2 & (k_1 + k_2)L^2/4 \end{bmatrix}$$

(a)

(b) $u_t = 1, u_\theta = 0$

(c) $u_t = 0, u_\theta = 1$

(d) $\ddot{u}_t = 1, \ddot{u}_\theta = 0$

(e) Inertia forces $= m/L$

(f) $\ddot{u}_t = 0, \ddot{u}_\theta = 1$

(g) Inertia forces $= -(m/L)x$

Figure E9.3
Observe that now the stiffness matrix has coupling terms because the chosen DOFs are not the displacements at the locations of the springs.

2. Determine the mass matrix. Impart a unit acceleration \( \ddot{u}_t = 1 \) with \( \ddot{u}_\theta = 0 \), determine the acceleration distribution (Fig. E9.3d) and the associated inertia forces, and identify \( m_t \) and \( m_\theta \) (Fig. E9.3e). By statics, \( m_t = m \) and \( m_\theta = 0 \). Now impart a unit rotational acceleration \( \ddot{u}_\theta = 1 \) with \( \ddot{u}_t = 0 \), determine the resulting accelerations (Fig. E9.3f) and the associated inertia forces, and identify \( m_{t\theta} \) and \( m_{\theta\theta} \) (Fig. E9.3g). By statics, \( m_{t\theta} = 0 \) and \( m_{\theta\theta} = mL^2/12 \). Note that \( m_{\theta\theta} = I_O \), the moment of inertia of the bar about an axis that passes through \( O \) and is perpendicular to the plane of rotation. Thus the mass matrix is

\[
\mathbf{\tilde{m}} = \begin{bmatrix} m & 0 \\ 0 & mL^2/12 \end{bmatrix}
\]  

(b)

Now the mass matrix is diagonal (i.e., it has no coupling terms) because the DOFs of this rigid bar are defined at the mass center.

3. Determine the equations of motion. Substituting \( \mathbf{u} = \langle u_t \hspace{1em} u_\theta \rangle^T, \mathbf{p} = \langle p_t \hspace{1em} p_\theta \rangle^T \), and Eqs. (a) and (b) in Eq. (9.2.12) gives

\[
\begin{bmatrix} m & 0 \\ 0 & mL^2/12 \end{bmatrix} \begin{bmatrix} \ddot{u}_t \\ \ddot{u}_\theta \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & (k_2 - k_1)L/2 \\ (k_2 - k_1)L/2 & (k_1 + k_2)L^2/4 \end{bmatrix} \begin{bmatrix} u_t \\ u_\theta \end{bmatrix} = \begin{bmatrix} p_t \\ p_\theta \end{bmatrix}
\]

(c)

The two differential equations are now coupled through the stiffness matrix.

We should note that if the equations of motion for a system are available in one set of DOFs, they can be transformed to a different choice of DOF. This concept is illustrated for the system of Fig. E9.2a. Suppose that the mass and stiffness matrices and the applied force vector for the system are available for the first choice of DOF, \( \mathbf{u} = \langle u_1 \hspace{1em} u_2 \rangle^T \). These displacements are related to the second set of DOF, \( \mathbf{\tilde{u}} = \langle u_t \hspace{1em} u_\theta \rangle^T \), by

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -L/2 \\ 1 & L/2 \end{bmatrix} \begin{bmatrix} u_t \\ u_\theta \end{bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{a}\mathbf{\tilde{u}}
\]

(d)

where \( \mathbf{a} \) denotes the coordinate transformation matrix. The stiffness and mass matrices and the applied force vector for the \( \mathbf{\tilde{u}} \) DOFs are given by

\[
\begin{align*}
\mathbf{\tilde{k}} &= \mathbf{a}^T \mathbf{k} \mathbf{a} \\
\mathbf{\tilde{m}} &= \mathbf{a}^T \mathbf{m} \mathbf{a} \\
\mathbf{\tilde{p}} &= \mathbf{a}^T \mathbf{p}
\end{align*}
\]

(e)

Substituting for \( \mathbf{a} \) from Eq. (d) and for \( \mathbf{k}, \mathbf{m}, \) and \( \mathbf{p} \) from Example 9.2 into Eq. (e) leads to \( \mathbf{\tilde{k}} \) and \( \mathbf{\tilde{m}} \), which are identical to Eqs. (a) and (b) and to the \( \mathbf{\tilde{p}} \) in Eq. (c).
Example 9.4

A massless cantilever beam of length \( L \) supports two lumped masses \( mL/2 \) and \( mL/4 \) at the midpoint and free end as shown in Fig. E9.4a. The flexural rigidity of the uniform beam is \( EI \). With the four DOFs chosen as shown in Fig. E9.4b and the applied forces \( p_1(t) \) and \( p_2(t) \), formulate the equations of motion of the system. Axial and shear deformations in the beam are neglected.

Solution

The beam consists of two beam elements and three nodes. The left node is constrained and each of the other two nodes has two DOFs (Fig. E9.4b). Thus, the displacement vector \( \mathbf{u} = (u_1 \ u_2 \ u_3 \ u_4)^T \).

1. Determine the mass matrix. With the DOFs defined at the locations of the lumped masses, the diagonal mass matrix is given by Eq. (9.2.10):

\[
\mathbf{m} = \begin{bmatrix}
\frac{mL}{4} & 0 \\
0 & \frac{mL}{2} \\
0 & 0
\end{bmatrix}
\]

2. Determine the stiffness matrix. Several methods are available to determine the stiffness matrix. Here we use the direct equilibrium method based on the definition of stiffness influence coefficients (Appendix 1).

To obtain the first column of the stiffness matrix, we impose \( u_1 = 1 \) and \( u_2 = u_3 = u_4 = 0 \). The stiffness influence coefficients are \( k_{11} \) (Fig. E9.4c). The forces necessary at the nodes of each beam element to maintain the deflected shape are determined from the beam stiffness coefficients (Fig. E9.4d). The two sets of forces in figures (c) and (d) are one and the same. Thus \( k_{11} = 96EI/L^3 \), \( k_{21} = -96EI/L^3 \), \( k_{31} = -24EI/L^2 \), and \( k_{41} = -24EI/L^2 \).

The second column of the stiffness matrix is obtained in a similar manner by imposing \( u_2 = 1 \) with \( u_1 = u_3 = u_4 = 0 \). The stiffness influence coefficients are \( k_{12} \) (Fig. E9.4e) and the forces on each beam element necessary to maintain the imposed displacements are shown in Fig. E9.4f. The two sets of forces in figures (e) and (f) are one and the same. Thus \( k_{12} = -96EI/L^3 \), \( k_{22} = 24EI/L^2 \), \( k_{32} = 96EI/L^3 + 96EI/L^3 = 192EI/L^3 \), and \( k_{42} = -24EI/L^2 + 24EI/L^2 = 0 \).

The third column of the stiffness matrix is obtained in a similar manner by imposing \( u_3 = 1 \) with \( u_1 = u_2 = u_4 = 0 \). The stiffness influence coefficients \( k_{13} \) are shown in Fig. E9.4g and the nodal forces in Fig. E9.4h. Thus \( k_{13} = -24EI/L^2 \), \( k_{23} = 24EI/L^2 \), \( k_{33} = 8EI/L \), and \( k_{43} = 4EI/L \).

The fourth column of the stiffness matrix is obtained in a similar manner by imposing \( u_4 = 1 \) with \( u_1 = u_2 = u_3 = 0 \). The stiffness influence coefficients \( k_{14} \) are shown in Fig. E9.4i, and the nodal forces in Fig. E9.4j. Thus \( k_{14} = -24EI/L^2 \), \( k_{34} = 4EI/L \), \( k_{24} = -24EI/L^2 + 24EI/L^2 = 0 \), and \( k_{44} = 8EI/L + 8EI/L = 16EI/L \).

With all the stiffness influence coefficients determined, the stiffness matrix is

\[
\mathbf{k} = \frac{8EI}{L^3} \begin{bmatrix}
12 & -12 & -3L & -3L \\
-12 & 24 & 3L & 0 \\
-3L & 3L & L^2 & L^2/2 \\
-3L & 0 & L^2/2 & 2L^2
\end{bmatrix}
\]

3. Determine the equations of motion. The governing equations are

\[
\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)
\]

where \( \mathbf{m} \) and \( \mathbf{k} \) are given by Eqs. (a) and (b), and \( \mathbf{p}(t) = (p_1(t) \ p_2(t) \ 0 \ 0)^T \).
Figure E9.4
Example 9.5

Derive the equations of motion of the beam of Example 9.4 (also shown in Fig. E9.5a) expressed in terms of the displacements $u_1$ and $u_2$ of the masses (Fig. E9.5b).

**Solution**  This system is the same as that in Example 9.4, but its equations of motion will be formulated considering only the translational DOFs $u_1$ and $u_2$ (i.e., the rotational DOFs $u_3$ and $u_4$ will be excluded).

1. **Determine the stiffness matrix.** In a statically determinate structure such as the one in Fig. E9.5a, it is usually easier to calculate first the flexibility matrix and invert it to obtain

\[
f_{S1} = 1, f_{S2} = 0
\]

\[
f_{S2} = 1, f_{S1} = 0
\]

\[
f_{S1} = 1
\]

\[
f_{S2} = 1
\]

\[
f_{f_{11}}
\]

\[
f_{f_{11}}
\]

\[
f_{f_{22}}
\]

\[
f_{f_{22}}
\]

\[
\begin{bmatrix}
  16 & 5 \\
  5 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  2 & -5 \\
  -5 & 16
\end{bmatrix}
\]

\[
\begin{bmatrix}
  mL/4 \\
  mL/2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  p_1(t) \\
  p_2(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  mL/4 \\
  mL/2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  48EI \\
  7L^3
\end{bmatrix}
\]

The off-diagonal elements $\hat{f}_{12}$ and $\hat{f}_{21}$ are equal, as expected, because of Maxwell’s theorem of reciprocal deflections. By inverting $\hat{f}$, the stiffness matrix is obtained:

\[
\kappa = \frac{48EI}{7L^3} \begin{bmatrix}
  2 & -5 \\
  -5 & 16
\end{bmatrix}
\]

2. **Determine the mass matrix.** This is a diagonal matrix because the lumped masses are located where the DOFs are defined:

\[
\mathbf{m} = \begin{bmatrix}
  mL/4 \\
  mL/2
\end{bmatrix}
\]

3. **Determine the equations of motion.** Substituting $\mathbf{m}$, $\kappa$, and $\mathbf{p}(t) = \begin{bmatrix} p_1(t) & p_2(t) \end{bmatrix}^T$ in Eq. (9.2.12) with $c = 0$ gives

\[
\begin{bmatrix}
  mL/4 \\
  mL/2
\end{bmatrix} \{\ddot{u}_1\} + \frac{48EI}{7L^3} \begin{bmatrix}
  2 & -5 \\
  -5 & 16
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  p_1(t) \\
  p_2(t)
\end{bmatrix}
\]
Example 9.6

Formulate the free vibration equations for the two-element frame of Fig. E9.6a. For both elements the flexural stiffness is $EI$, and axial deformations are to be neglected. The frame is massless with lumped masses at the two nodes as shown.

**Solution** The two degrees of freedom of the frame are shown. The mass matrix is

$$
\mathbf{m} = \begin{bmatrix}
3m \\
m
\end{bmatrix}
$$

(a)

Note that the mass corresponding to $\ddot{u}_1 = 1$ is $2m + m = 3m$ because both masses will undergo the same acceleration since the beam connecting the two masses is axially inextensible.

The stiffness matrix is formulated by first evaluating the flexibility matrix and then inverting it. The flexibility influence coefficients are identified in Fig. E9.6b and c, and the deflections are computed by standard procedures of structural analysis to obtain the flexibility matrix:

$$
\hat{\mathbf{f}} = \frac{L^3}{6EI} \begin{bmatrix}
2 & 3 \\
3 & 8
\end{bmatrix}
$$

This matrix is inverted to determine the stiffness matrix:

$$
\mathbf{k} = \frac{6EI}{7L^3} \begin{bmatrix}
8 & -3 \\
-3 & 2
\end{bmatrix}
$$

Thus the equations in free vibration of the system (without damping) are

$$
\begin{bmatrix}
3m & m \\
m & m
\end{bmatrix} \begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2
\end{bmatrix} + \frac{6EI}{7L^3} \begin{bmatrix}
8 & -3 \\
-3 & 2
\end{bmatrix} \begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
$$
Example 9.7

Formulate the equations of motion for the two-story frame in Fig. E9.7a. The flexural rigidity of the beams and columns and the lumped masses at the floor levels are as noted. The dynamic excitation consists of lateral forces $p_1(t)$ and $p_2(t)$ at the two floor levels. The story height is $h$ and the bay width $2h$. Neglect axial deformations in the beams and the columns.

\[ E I \]

\[ f_i = \begin{cases} 1 & \text{for } i = 1, 5, 6 \\ 2 & \text{for } i = 2 \\ 2m & \text{for } i = 3, 4 \end{cases} \]

\[ k_i = \begin{cases} k_{31} & \text{for } i = 1 \\ k_{11} & \text{for } i = 5 \\ k_{41} & \text{for } i = 6 \\ k_{21} & \text{for } i = 2 \\ k_{12} & \text{for } i = 3 \\ k_{23} & \text{for } i = 4 \\ k_{33} & \text{for } i = 5 \\ k_{43} & \text{for } i = 6 \\ k_{53} & \text{for } i = 7 \\ k_{63} & \text{for } i = 8 \end{cases} \]

\[ L = 2h \]

\[ (a) \]

\[ (b) \]

\[ (c) \]

Figure E9.7

Solution  The system has six degrees of freedom shown in Fig. E9.7a: lateral displacements $u_1$ and $u_2$ of the floors and joint rotations $u_3$, $u_4$, $u_5$, and $u_6$. The displacement vector is

\[ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \]

The mass matrix is given by Eq. (9.2.10):

\[ \mathbf{m} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \]

The stiffness influence coefficients are evaluated following the procedure of Example 9.4. A unit displacement is imposed, one at a time, in each DOF while constraining the other five DOFs, and the stiffness influence coefficients (e.g., shown in Fig. E9.7b and c for $u_1 = 1$ and $u_3 = 1$, respectively) are calculated by statics from the nodal forces for individual structural elements associated with the imposed displacements. These nodal forces are determined from the beam stiffness coefficients (Appendix 1). The result is

\[ \mathbf{k} = \frac{EI}{h^3} \begin{bmatrix} 72 & -24 & 6h & 6h & -6h & -6h \\ -24 & 24 & 6h & 6h & 6h & 6h \\ 6h & 6h & 16h^2 & 2h^2 & 2h^2 & 0 \\ 6h & 6h & 2h^2 & 16h^2 & 0 & 2h^2 \\ -6h & 6h & 2h^2 & 0 & 6h^2 & h^2 \\ -6h & 6h & 0 & 2h^2 & h^2 & 6h^2 \end{bmatrix} \]

The dynamic forces applied are lateral forces $p_1(t)$ and $p_2(t)$ at the two floors without any moments at the nodes. Thus the applied force vector is

\[ \mathbf{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \]

The equations of motion are

\[ \mathbf{m} \ddot{\mathbf{u}} + \mathbf{k} \mathbf{u} = \mathbf{p}(t) \]

where $\mathbf{u}$, $\mathbf{m}$, $\mathbf{k}$, and $\mathbf{p}(t)$ are given by Eqs. (a), (b), (c), and (d), respectively.
Static Condensation

Static condensation is a method to exclude the DOFs with no force from dynamic analysis. Typically the formulation of stiffness matrix in static analysis considers all unrestrained DOFs at joints between structural members. Some of DOFs may not be associated with any mass in dynamic analysis, for example, rotation DOFs in a lumped-mass model, so they should be excluded to simplify the dynamic analysis.

The equations of motion for a building shown above is

\[
\begin{bmatrix}
    m_{tt} & 0 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    \ddot{u}_t \\
    \ddot{u}_o
\end{bmatrix} +
\begin{bmatrix}
    k_{tt} & k_{to} \\
    k_{ot} & k_{oo}
\end{bmatrix}
\begin{bmatrix}
    u_t \\
    u_o
\end{bmatrix} =
\begin{bmatrix}
    p_t(t) \\
    0
\end{bmatrix}
\]

It is partitioned into translation \((u_t)\) and rotation \((u_o)\) DOFs. Each part involves vectors and sub-matrices.

Each group of partitioned equations are

\[m_{tt}\ddot{u}_t + k_{tt}u_t + k_{to}u_o = p_t(t)\quad\text{and}\quad k_{ot}u_t + k_{oo}u_o = 0\]
Because no inertia terms and external forces are associated with the rotations, \( \mathbf{u}_o \) can be solved:

\[
\mathbf{u}_o = -\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}\mathbf{u}_t
\]

Then, we can substitute \( \mathbf{u}_o \) into the equation for translational DOFs and obtain equations of motion which are simpler as they involve only translation DOFs.

\[
\mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \hat{\mathbf{k}}_{tt}\mathbf{u}_t = \mathbf{p}_t(t)
\]

where the condensed stiffness matrix is

\[
\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{to}^T\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}
\]

Note that \( \mathbf{k}_{to} = \mathbf{k}_{ot}^T \) because \( \mathbf{k} \) is a symmetric matrix.
Example 9.8

Examples 9.4 and 9.5 were concerned with formulating the equations of motion for a cantilever beam with two lumped masses. The degrees of freedom chosen in Example 9.5 were the translational displacements \( u_1 \) and \( u_2 \) at the lumped masses; in Example 9.4 the four DOFs were \( u_1, u_2, u_3 \), and node rotations \( u_3 \) and \( u_4 \). Starting with the equations governing these four DOFs, derive the equations of motion in the two translational DOFs.

**Solution** The vector of four DOFs is partitioned in two parts: \( \mathbf{u}_t = (u_1 \quad u_2)^T \) and \( \mathbf{u}_0 = (u_3 \quad u_4)^T \). The equations of motion governing \( \mathbf{u}_t \) are given by Eq. (9.3.4), where

\[
\mathbf{m}_t = \begin{bmatrix}
\frac{mL}{4} \\
\frac{mL}{2}
\end{bmatrix}, \quad \mathbf{p}_t(t) = \begin{bmatrix}
p_1(t) \\
p_2(t)
\end{bmatrix}^T
\]

(a)

To determine \( \mathbf{k}_{tt} \), the 4 × 4 stiffness matrix determined in Example 9.4 is partitioned:

\[
\mathbf{k} = \begin{bmatrix}
\mathbf{k}_{tt} & \mathbf{k}_{t0} \\
\mathbf{k}_{0t} & \mathbf{k}_{00}
\end{bmatrix} = \frac{8EI}{L^3} \begin{bmatrix}
12 & -12 & -3L & -3L \\
-12 & 24 & 3L & 0 \\
-3L & 3L & L^2 & L^2/2 \\
-3L & 0 & L^2/2 & 2L^2
\end{bmatrix}
\]

(b)

Substituting these submatrices in Eq. (9.3.5) gives the condensed stiffness matrix:

\[
\mathbf{k}_{tt} = \frac{48EI}{7L^3} \begin{bmatrix}
2 & -5 \\
-5 & 16
\end{bmatrix}
\]

(c)

This stiffness matrix of Eq. (c) is the same as that obtained in Example 9.5 by inverting the flexibility matrix corresponding to the two translational DOFs.

Substituting the stiffness submatrices in Eq. (9.3.3) gives the relation between the condensed DOF \( \mathbf{u}_0 \) and the dynamic DOF \( \mathbf{u}_t \):

\[
\mathbf{u}_0 = \mathbf{T} \mathbf{u}_t, \quad \mathbf{T} = \frac{1}{L} \begin{bmatrix}
2.57 & -3.43 \\
0.857 & 0.857
\end{bmatrix}
\]

(d)

The equations of motion are given by Eq. (9.3.4), where \( \mathbf{m}_t \) and \( \mathbf{p}_t(t) \) are defined in Eq. (a) and \( \mathbf{k}_{tt} \) in Eq. (c). These are the same as Eq. (c) of Example 9.5.
Equation of motion:
Planar systems subjected to translational ground motion

At each instant of time, displacement of each mass is

\[ u'_j(t) = u_j(t) + u_g(t) \]

For \( N \) masses, the displacements can be written in compact form as a vector.

\[ \mathbf{u}'(t) = \mathbf{u}(t) + u_g(t) \mathbf{1} \]

where \( \mathbf{1} \) is a vector of order \( N \) with each element equal to unity.

Figure 9.4.1  (a) Building frame; (b) tower.

The equations of motion previously derived for a MDF system subjected to external force \( p(t) \) is still valid except that the external force for this case (ground excitation) is zero.

\[ \mathbf{f}_l + \mathbf{f}_D + \mathbf{f}_S = \mathbf{0} \]

Only relative displacements \( \mathbf{u} \) between masses and the base produce deformation and elastic and damping forces.
The inertia forces $f_i$ are related to the total acceleration $\ddot{u}'$.

$$f_i = m\ddot{u}'$$

Substituting $f_i = m\ddot{u}'$ in the equilibrium equation, we get

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t)$$

The right hand side is the effective earthquake forces due to ground motion excitation.

$$p_{eff}(t) = -m\ddot{u}_g(t)$$

This is valid when a unit ground displacement results in a unit total displacement of all DOFs. In general, this is not always the case. We introduce the influence vector $\iota$ to represent the influence of ground displacement on total displacement at DOFs.

$$u'(t) = u_g(t) + u(t)$$

The equations of motion are

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t)$$
For example, vertical DOF $u_3$ is not displaced when the ground moves horizontally. The influence vector is

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The effective earthquake force is

$$\mathbf{p}_{\text{eff}}(t) = -m\ddot{u}_g(t) = -\ddot{u}_g(t)\begin{bmatrix} m_1 \\ m_2 + m_3 \\ m_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\ddot{u}_g(t)\begin{bmatrix} m_1 \\ m_2 + m_3 \\ 0 \end{bmatrix}$$

Note that the mass corresponding to $\ddot{u}_2 = 1$ is $m_2 + m_3$ because both masses will undergo the same acceleration since the connecting beam is axially rigid.

The effective earthquake force is zero in the vertical DOF because the ground motion is horizontal.
Inelastic systems

For inelastic systems, the force resisting deformation is no longer linear relationship and is described by a nonlinear function

\[ f_s = f_s(u, \dot{u}) \]

The equation of motion becomes

\[ m \ddot{u} + c \dot{u} + f_s(u, \dot{u}) = -mu_t(t) \]

Such equation has to be solved by numerical methods as presented in Chapter 5.

Problem statement

Given a system with known, mass matrix \( m \), damping matrix \( c \), stiffness matrix \( k \), and excitation \( p(t) \) or \( u_t(t) \), we want to determine the response of the system.

Response can be any response quantity such as displacement \( u(t) \), velocity, acceleration of masses or internal forces, which is closely related to the relative displacement.

By the concept of equivalent static force, internal forces can be obtained by static analysis of structure subjected to a set of equivalent static forces

\[ f_s(t) = ku(t) \]