NON-PRISMATIC MEMBERS

Usually the structural members that we are familiar with are prismatic, meaning the cross section is constant throughout the length of the member.

$EI$, $EA$ and $GA$ are constant, so it is easier to analyze such structural member as we can conveniently use all formula derived for the prismatic members.

In practice, it may be more economical to reduce the cross section of the member where the bending moment is smaller and enlarge the section where the bending moment is larger.

Sometimes, it is necessary to reduce to depth of the beam, called haunched beam, to allow mechanical ducts to pass under, or to gain more clear height for doorway.

If the non-prismatic member comprises of connecting prismatic sub-elements, we can simply analyze the non-prismatic element by dividing it into several prismatic elements.

If the non-prismatic member is part of a larger structure, the displacement method, e.g. direct stiffness method, can still be used to analyze the structure, but the stiffness of the non-prismatic member has to be obtained in a special way.
It can be calculated from the inverse of the flexibility of the member using virtual work method, or by column-analogy method. Only the first approach will be presented here.

The derivation of flexibility and stiffness matrices of non-prismatic members will be presented in a general form, which also applies to the prismatic members. The particular case of prismatic member will be shown for reference.

**AXIAL FORCE ELEMENT**

Flexibility is the deformation due to a unit force, which can be determined by the virtual work method (virtual force):

1) Apply a unit virtual force and
2) Equate the external virtual work to internal virtual work.

\[
\delta = \int \int \delta N \cdot dx \cdot EA(x) \cdot dx
\]

For a prismatic member, \( EA \) is constant. The flexibility is

\[
f = \int_{0}^{L} \frac{N}{EA(x)} \cdot \delta N \cdot dx = \int_{0}^{L} \frac{1}{EA(x)} dx \quad \text{where} \quad N = \delta N = 1
\]
\[ f = \int_{0}^{L} \frac{1}{EA} \, dx = \frac{L}{EA} \]

Stiffness is the reciprocal of flexibility,

\[ k = \frac{1}{f} = \frac{EA}{L} \]

If the member is non-prismatic, \( EA \) is not constant but varies along the length of the member.

Flexibility is

\[ f = \int_{0}^{L} \frac{N}{EA(x)} \cdot \delta N \cdot dx = \int_{0}^{L} \frac{1}{EA(x)} \, dx \]

Stiffness is

\[ k = \frac{1}{f} = \left[ \int_{0}^{L} \frac{1}{EA(x)} \, dx \right]^{-1} \]
Example

A tapered rod has the cross-sectional area, \( A = A_0 \) at one end and it decreases to \( A_0(1-m) \) at the other end.

\[
A(x) = A_0 \left(1 - \frac{mx}{L}\right)
\]

Flexibility of the member is

\[
f = \int_0^L \frac{N}{EA(x)} \cdot \delta N \cdot dx = \int_0^L \frac{1}{EA_0 \left(1 - \frac{mx}{L}\right)} dx
\]

\[
f = \frac{-L}{mEA_0} \int_0^L \frac{1}{1 - \frac{mx}{L}} \cdot d \left(1 - \frac{mx}{L}\right)
\]

\[
= \frac{-L}{mEA_0} \left[\ln \left(1 - \frac{mx}{L}\right)\right]_{x=0}^{x=L}
\]

\[
= \frac{-L}{mEA_0} \ln(1 - m)
\]

Stiffness is

\[
k = \frac{1}{f} = \frac{-m}{\ln(1 - m)} \frac{EA_0}{L}
\]

Let say \( m=0.5 \),

\[
\frac{-m}{\ln(1 - m)} = \frac{-0.5}{\ln(1 - 0.5)} = 0.721.
\]

The stiffness will be \( k = 0.721 \frac{EA_0}{L} \).
BEAM ELEMENT

The deformation modes of a basic 2-D beam element involve only end-rotations relative to the cord if the axial deformation is neglected.

Flexibility matrix of the basic system is

\[
\begin{bmatrix}
\int_0^L \frac{m_1(x) \cdot m_1(x)}{EI(x)} \, dx & \int_0^L \frac{m_1(x) \cdot m_2(x)}{EI(x)} \, dx \\
\int_0^L \frac{m_1(x) \cdot m_2(x)}{EI(x)} \, dx & \int_0^L \frac{m_2(x) \cdot m_2(x)}{EI(x)} \, dx
\end{bmatrix}
\]

For a prismatic beam, \( EI \) is constant. Flexibility matrix is

\[
f = \frac{1}{EI} \int_{x=0}^{x=L} \begin{bmatrix}
\left(-1 + \frac{x}{L}\right)^2 & \left(-1 + \frac{x}{L}\right) \frac{x}{L} \\
\left(-1 + \frac{x}{L}\right) \frac{x}{L} & \left(-1 + \frac{x}{L}\right)^2
\end{bmatrix} \, dx = \frac{L}{EI} \begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix}
\]
For a prismatic beam, the stiffness matrix of the basic system is

\[
k = f^{-1} = \frac{EI}{L} \begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix}^{-1} = \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}
\]

The stiffness matrix of basic deformation can then be transformed to the stiffness matrix of 2-D beam in local coordinates with 4 DOFs using matrix A, which is the relationship between local coordinate, D, and basic deformation, d.

\[
d = AD
\]

\[
A = \begin{bmatrix} 1/L & 1 & -1/L & 0 \\ 1/L & 0 & -1/L & 1 \end{bmatrix}
\]

\[
d_1 \quad d_2
\]

\[
D_1 \quad D_2 \quad D_3 \quad D_4
\]

\[
1/L \quad 1
\]

\[
1 \quad 1/L
\]
The element force in the basic deformation mode is

\[ q = kd \]

The element force in the local coordinate is

\[ Q = A^T q = A^T kd = A^T kAD = KD \]

\[
K = A^T kA = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2 \\
\end{bmatrix}
\]

This element stiffness \( K \) in local coordinated needs to be rotated to global coordinate before assembled for the whole structure by direct stiffness method.
For a non-prismatic beam, the flexibility matrix is

\[
\mathbf{f} = \begin{bmatrix}
    f_{11} & f_{12} \\
    f_{21} & f_{22}
\end{bmatrix} = 
\begin{bmatrix}
    \int_0^L \frac{m_1(x) \cdot m_1(x)}{EI(x)} \, dx & \int_0^L \frac{m_1(x) \cdot m_2(x)}{EI(x)} \, dx \\
    \int_0^L \frac{m_1(x) \cdot m_2(x)}{EI(x)} \, dx & \int_0^L \frac{m_2(x) \cdot m_2(x)}{EI(x)} \, dx
\end{bmatrix}
\]

where \( m_1(x) = -1 + \frac{x}{L} \) and \( m_2(x) = \frac{x}{L} \)

For a non-prismatic beam, the stiffness matrix is

\[
\mathbf{k} = \mathbf{f}^{-1} = \begin{bmatrix}
    k_{11} & k_{12} \\
    k_{21} & k_{22}
\end{bmatrix} = \frac{1}{f_{11}f_{22} - f_{12}f_{21}} \begin{bmatrix}
    f_{22} & -f_{21} \\
    -f_{12} & f_{11}
\end{bmatrix}
\]

\( k_{11} \) and \( k_{22} \) are rotational stiffness at node 1 and 2, respectively.

The ratio \( C_{12} = \frac{k_{12}}{k_{11}} \) is called Carryover factor from node 1 to 2

The ratio \( C_{21} = \frac{k_{12}}{k_{22}} \) is called Carryover factor from node 2 to 1
Example

Find stiffness matrix of this beam.

\[
\begin{align*}
\mathbf{f}_{11} & = \int_0^L \frac{1}{EI(x)} \left( -1 + \frac{x}{L} \right)^2 dx \\
& = \int_0^{0.2L} \frac{1}{1.5EI} \left( -1 + \frac{x}{L} \right)^2 dx + \int_{0.2L}^{0.8L} \frac{1}{EI} \left( -1 + \frac{x}{L} \right)^2 dx + \int_{0.8L}^L \frac{1}{1.5EI} \left( -1 + \frac{x}{L} \right)^2 dx \\
& = 0.2782 \frac{L}{EI}
\end{align*}
\]
\[ f_{22} = \int_0^L \frac{1}{EI(x)} \left( \frac{x}{L} \right)^2 \, dx \]

\[ = \int_0^{0.2L} \frac{1}{1.5EI} \left( \frac{x}{L} \right)^2 \, dx + \int_{0.2L}^{0.8L} \frac{1}{EI} \left( \frac{x}{L} \right)^2 \, dx + \int_{0.8L}^L \frac{1}{1.5EI} \left( \frac{x}{L} \right)^2 \, dx \]

\[ = 0.2782 \frac{L}{EI} \]

\[ f_{12} = f_{21} = \int_0^L \frac{1}{EI(x)} \left( -1 + \frac{x}{L} \right) \left( \frac{x}{L} \right) \, dx \]

\[ = \int_0^{0.2L} \frac{1}{1.5EI} \left( -1 + \frac{x}{L} \right) \left( \frac{x}{L} \right) \, dx + \int_{0.2L}^{0.8L} \frac{1}{EI} \left( -1 + \frac{x}{L} \right) \left( \frac{x}{L} \right) \, dx + \int_{0.8L}^L \frac{1}{1.5EI} \left( -1 + \frac{x}{L} \right) \left( \frac{x}{L} \right) \, dx \]

\[ = -0.1551 \frac{L}{EI} \]

\[ \mathbf{f} = \frac{L}{EI} \begin{bmatrix} 0.2782 & -0.1551 \\ -0.1551 & 0.2782 \end{bmatrix} \quad \mathbf{k} = \frac{EI}{L} \begin{bmatrix} 5.215 & 2.908 \\ 2.908 & 5.215 \end{bmatrix} \]
FIXED-END MOMENT (FEM)

If the load is applied within the element, for example, concentrated load (point load) or distributed load, the load has to be converted to equivalent fixed-end moment before applying to the joint in the direct stiffness method.

Use flexibility method to solve for moments at fixed support by choosing those moments to be the redundant.

The compatibility condition requires that end rotations are zero.

\[ \theta_1 = \theta_1^o + f_{11} \text{FEM}_1 + f_{12} \text{FEM}_2 = 0 \]

\[ \theta_2 = \theta_2^o + f_{21} \text{FEM}_1 + f_{22} \text{FEM}_2 = 0 \]
where the end rotations of the primary structure due to external load are

\[
\theta_1^o = \int_0^L \frac{M^o(x)m_1(x)}{EI(x)} \, dx \quad \text{and} \quad \theta_2^o = \int_0^L \frac{M^o(x)m_2(x)}{EI(x)} \, dx
\]

and the flexibility coefficients were given earlier.

Therefore, we can solve for FEMS from

\[
\begin{bmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{bmatrix}
\begin{bmatrix}
  \text{FEM}_1 \\
  \text{FEM}_2
\end{bmatrix} = -
\begin{bmatrix}
  \theta_1^o \\
  \theta_2^o
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \text{FEM}_1 \\
  \text{FEM}_2
\end{bmatrix} = -
\begin{bmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
  \theta_1^o \\
  \theta_2^o
\end{bmatrix} = -K
\begin{bmatrix}
  \theta_1^o \\
  \theta_2^o
\end{bmatrix}
\]

When the global structure is analyzed, the effect of element load is considered as external moments equal to \(-\text{FEM}\) applied at the joint at both ends of the element.
Example Find the fixed-end moments due to a uniform distributed load.

\[
\theta^o_x = \int_0^L \frac{1}{EI(x)} M^o(x) \left(-1 + \frac{x}{L}\right) dx
\]

\[
= \int_0^{0.2L} \frac{1}{1.5EI} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right) \left(-1 + \frac{x}{L}\right) dx + \int_{0.2L}^{0.8L} \frac{1}{EI} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right) \left(-1 + \frac{x}{L}\right) dx + \int_{0.8L}^L \frac{1}{1.5EI} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right) \left(-1 + \frac{x}{L}\right) dx
\]

\[
= -0.03878 \frac{wL^3}{EI}
\]
\[
\theta_2^o = \int_0^L \frac{1}{EI(x)} M^o(x) \left( \frac{x}{L} \right) dx
\]

\[
= \int_0^{0.2L} \frac{1}{1.5EI} \left( \frac{wLx}{2} - \frac{wx^2}{2} \right) \left( \frac{x}{L} \right) dx + \int_{0.2L}^{0.8L} \frac{1}{EI} \left( \frac{wLx}{2} - \frac{wx^2}{2} \right) \left( \frac{x}{L} \right) dx + \int_{0.8L}^L \frac{1}{1.5EI} \left( \frac{wLx}{2} - \frac{wx^2}{2} \right) \left( \frac{x}{L} \right) dx
\]

\[
= 0.03878 \frac{wL^3}{EI}
\]

\[
\{ \text{FEM}_1 \} = -\frac{EI}{L} \begin{bmatrix} 5.215 & 2.908 \\ 2.908 & 5.215 \end{bmatrix} \begin{bmatrix} -0.03878 \\ +0.03878 \end{bmatrix} \frac{wL^3}{EI} = \begin{bmatrix} +0.0895 \\ -0.0895 \end{bmatrix} wL^2
\]

\[
\theta_1^o = -0.03878 \frac{wL^3}{EI} \quad \theta_2^o = 0.03878 \frac{wL^3}{EI} \quad \text{FEM}_1 = 0.0895wL^2 \quad \text{FEM}_2 = -0.0895wL^2
\]
**Example** Determine the bending moment at the middle support.

\[ w(x) = w \]

\[ \theta_1 \quad \theta_2 \quad \theta_3 \]

\[ w(x) = w \equiv 0.0895wL^2 \quad 0.0895wL^2 \]
\[
\frac{EI}{L} \begin{bmatrix} 5.215 & 2.908 & 0 \\ 2.908 & 5.215 + 5.215 & 2.908 \\ 0 & 2.908 & 5.215 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.0895 \\ 0.0895 - 0.0895 \\ +0.0895 \end{bmatrix} wL^2
\]

\[
\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -0.01716 \\ 0 \\ 0.01716 \end{bmatrix} \frac{wL^3}{EI}
\]

\[
\theta_1 = 0.01716 \frac{wL^3}{EI} \\
\theta_2 = 0 \\
\theta_3 = 0.01716 \frac{wL^3}{EI}
\]
\[ q = kd = \frac{EI}{L} \begin{bmatrix} 5.215 & 2.908 \\ 2.908 & 5.215 \end{bmatrix} \begin{bmatrix} -0.01716 \\ 0 \end{bmatrix} \frac{wL^3}{EI} = \begin{bmatrix} -0.0895 \\ -0.0500 \end{bmatrix} wL^2 \]

**Total moment acting on the member**

\[
0.0895wL^2 \\
\theta_1^\circ \\
\theta_2^\circ \\
0.0895wL^2 \\
0.05wL^2
\]

**Equivalent FEM due to element load**

\[
0.0895wL^2 \\
\theta_1^\circ \\
0.1395wL^2
\]

**Moment from adjacent element**