

2103-617: Advanced Dynamics
Handout # 2: Review of Dynamic Laws

1. **Newton's 2nd Law** (see Figure 1)

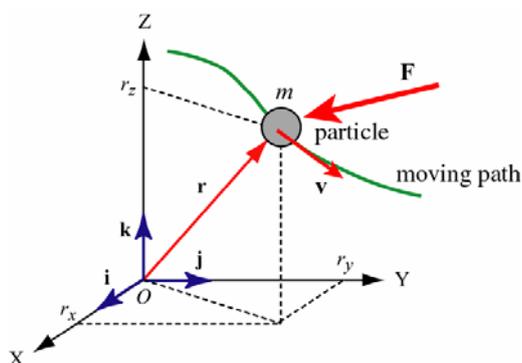


Figure 1: A particle moving in 3-D space

The Newton's 2nd Law actually states that *a particle acted upon by a resultant force \mathbf{F} moves so that the force is equal to time rate of change of linear momentum*. Or, i.e.:

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}, \quad (1)$$

where \mathbf{P} is the linear momentum defined by $\mathbf{P} = m\mathbf{v} = m\dot{\mathbf{r}}$. Note that \mathbf{r} and \mathbf{v} are the position vector and velocity of the particle, respectively. If the mass m is constant, then we got the familiar Newton's 2nd Law as

$$\mathbf{F} = \frac{d(m\dot{\mathbf{r}})}{dt} = m\ddot{\mathbf{r}} = m\mathbf{a}, \quad (2)$$

where \mathbf{a} is the absolute acceleration of the particle relative to the inertial reference frame.

As an example, let's derive the equation of motion of a pendulum as shown in the Figure 2.

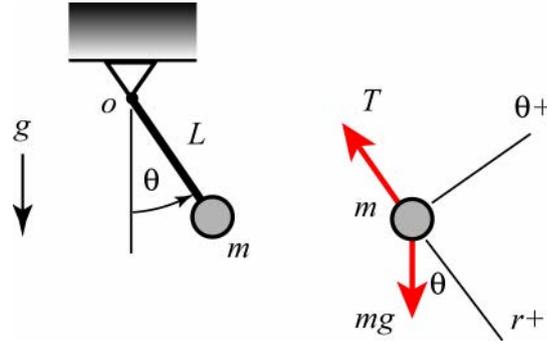


Figure 2: A pendulum

Kinematics relations: To completely describe the motion of the pendulum, we need only one degree of freedom. Let's choose θ as the coordinate to describe its motion. The acceleration components along the r and θ axes are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -L\dot{\theta}^2 \quad (3)$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = L\ddot{\theta} \quad (4)$$

Force equilibrium: From the Newton's 2nd Law and free body diagram in Figure 2, we obtain the governing equations:

$$[\sum F_r = ma_r] \quad mg \cos \theta - T = m(-L\dot{\theta}^2) \quad (5)$$

$$[\sum F_\theta = ma_\theta] \quad -mg \sin \theta = m(L\ddot{\theta}) \quad (6)$$

Rearrange equation (6), we get the **equation of motion** as

$$\ddot{\theta}(t) + \frac{g}{L} \sin \theta(t) = 0 \quad (7)$$

Note that equation (7) is the nonlinear ordinary differential equation which governs the motion of the pendulum for any arbitrary $\theta(t)$. In addition, this system has one degree of freedom, thus only one equation of motion is needed. Specifically, the number of equations of motion used to describe the motion is equal to the number of degrees of freedom of that system. If only the small oscillation about the equilibrium, such as about $\theta_0 = 0$, is considered, equation (7) can be linearized about this operating point. We will

determine such linearized equation in later example. Moreover, when the solution of (7) is obtained, the response $\theta(t)$ can be substituted into (5) to determine the tension T which is an internal dynamic force varying with time.

2. Moment and angular momentum

The moment of force \mathbf{F} about 'o', denoted by \mathbf{M}_o , is defined as

$$\mathbf{M}_o = \mathbf{r} \times \mathbf{F},$$

where \mathbf{r} is the position vector from point 'o' to the particle, as shown in Figure 1.

Angular momentum about 'o', denoted by \mathbf{H}_o , is defined as the moment of linear momentum \mathbf{P} , given by

$$\mathbf{H}_o = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times m\dot{\mathbf{r}}$$

The rate of change of angular momentum is then

$$\begin{aligned} \dot{\mathbf{H}}_o &= \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) \\ &= \mathbf{r} \times m\ddot{\mathbf{r}} + \dot{\mathbf{r}} \times m\dot{\mathbf{r}} \\ &= \mathbf{r} \times \mathbf{F} + \mathbf{0} \\ &= \mathbf{M}_o \end{aligned} \tag{8}$$

Equation (8) is the law of angular momentum, stating that the moment of force about the fixed point 'o' is equal to the rate of change of the angular momentum about that point.

Now let's derive the equation of motion for pendulum using the law of angular momentum.

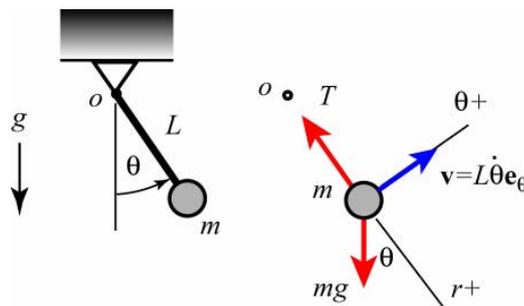


Figure 3: A pendulum

From Figure 3, the angular momentum of pendulum about 'o' is

$$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v} = L\mathbf{e}_r \times m(L\dot{\theta}\mathbf{e}_\theta) = mL^2\dot{\theta}\mathbf{k}$$

The rate of change of angular momentum of the pendulum is then

$$\dot{\mathbf{H}}_o = mL^2\ddot{\theta}\mathbf{k} \equiv I_o\ddot{\theta}\mathbf{k}$$

From the law of angular momentum, we can derive the equation of motion:

$$[\sum M_o = \dot{H}_o] \text{CCW+} \quad -mg(L \sin \theta) = mL^2\ddot{\theta}$$

Or

$$\ddot{\theta}(t) + \frac{g}{L} \sin \theta(t) = 0$$

3. **Dynamics of a system of particles** (not used directly, but applied to dynamics of a rigid body)

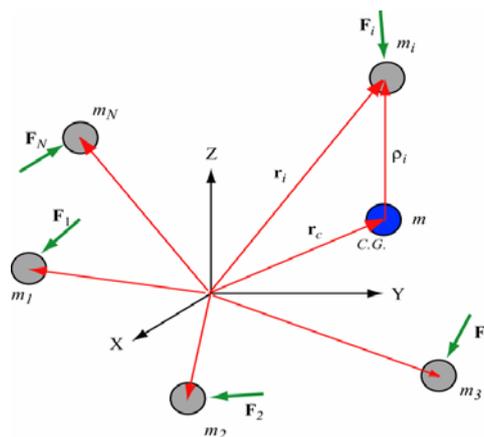


Figure 4: A system of particles

The Newton's 2nd Law can be applied to the system of particles as shown in Figure 4 as follows:

$$\sum \mathbf{F}_i = \sum_{i=1}^N m_i \mathbf{a}_i = m \mathbf{a}_c, \quad (9)$$

where $m = \sum_i m_i$ is the total mass, and \mathbf{a}_c is the acceleration of the mass center. The law

of angular momentum for the system of particles can be expressed as follows:

$$\sum \mathbf{M}_o = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = \dot{\mathbf{H}}_o, \quad (10)$$

where \mathbf{H}_o is the angular momentum about the fixed point o , given by $\mathbf{H}_o = \sum_{i=1}^N \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i$.

Alternatively the moment and angular momentum about the center of mass c can be derived as

$$\sum \mathbf{M}_c = \sum_{i=1}^N \boldsymbol{\rho}_i \times \mathbf{F}_i = \dot{\mathbf{H}}_c, \quad (11)$$

where \mathbf{H}_c is the angular momentum about the center of mass c , defined as

$$\mathbf{H}_c = \sum_{i=1}^N \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i.$$

4. Dynamics of a rigid body (for a 2-D or plane motion)

The Newton's 2nd Law and the law of angular momentum can be applied to a rigid body motion as described in the following summary.

4.1. Translation

Governing equation for the rigid body translation is

$$\mathbf{F} = m\mathbf{a} \quad (12)$$

Note that for the translation, the accelerations \mathbf{a} of any points on the rigid body are the same for the whole body.

4.2. Rotation about a fixed point 'o'

$$\mathbf{M}_o = \dot{\mathbf{H}}_o \quad (13)$$

For a rigid body, the angular momentum is simply

$$\mathbf{H}_o = \int \mathbf{r} \times \mathbf{v} dm = \int r^2 dm \omega \mathbf{k} = I_o \omega \mathbf{k}, \quad (14)$$

where ω is the angular velocity of the body, and I_o is the moment of inertia about o . Plug (14) into (13), the governing equation for the planar rotation of a rigid body in a scalar form is then

$$M_o = I_o \dot{\omega} = I_o \alpha \quad (15)$$

where M_o is the moment sum about the point of rotation o , and α is the angular acceleration of the body.

4.3. General plane motion (simultaneous translation and rotation)

The general planar motion of a rigid body is governed by two sets of equations.

1) Newton's 2nd Law (vector equation):

$$\mathbf{F} = m\mathbf{a}_c, \quad (16)$$

where \mathbf{a}_c is the acceleration of the mass center.

2) Law of angular momentum (scalar equation):

$$M_c = I_c \dot{\omega} = I_c \alpha, \quad (17)$$

where M_c is the moment sum about the center of mass c , I_c is the moment of inertia about the center of mass, and α is the angular acceleration of the body.