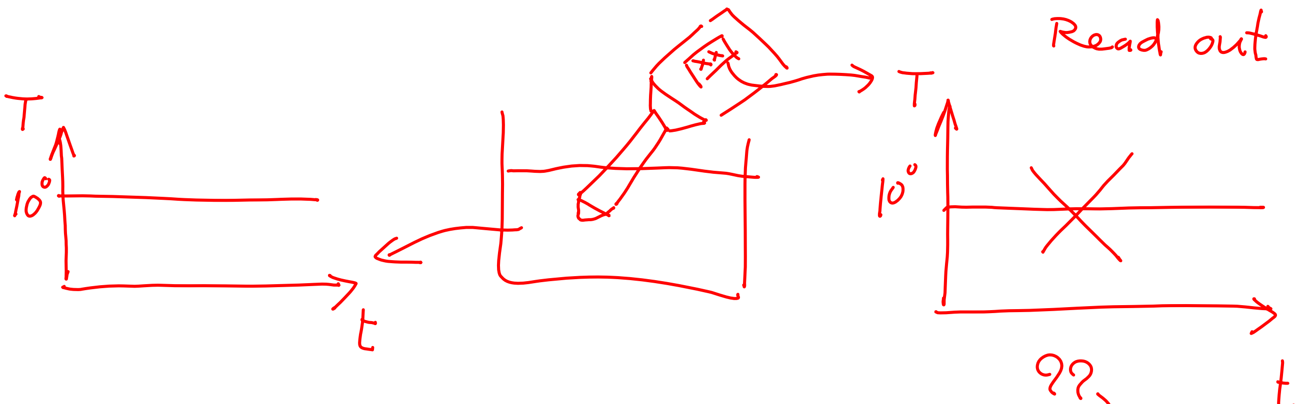
Dynamic Characteristics of Instruments and Measuring Systems

2103-602 Measurement and Instrumentation

Introduction

Instruments and measuring system *do not* respond to the change of input *instantly*.

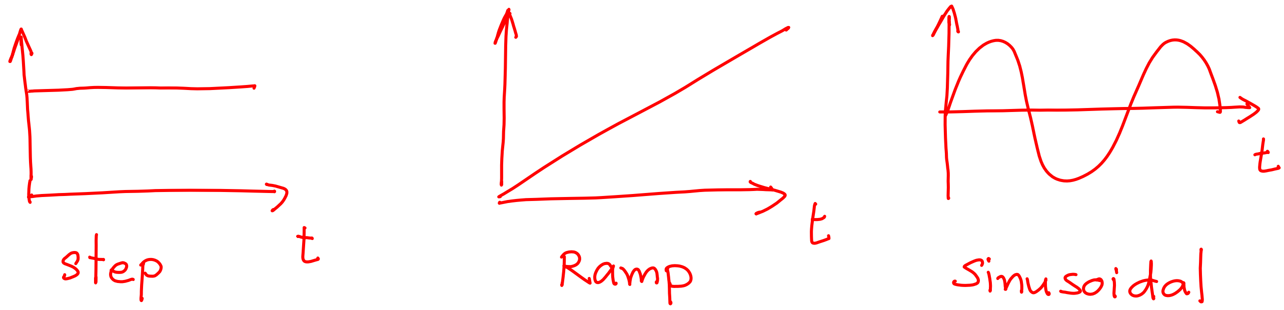


depend on dynamic characteristic

Introduction

Output response depends on:

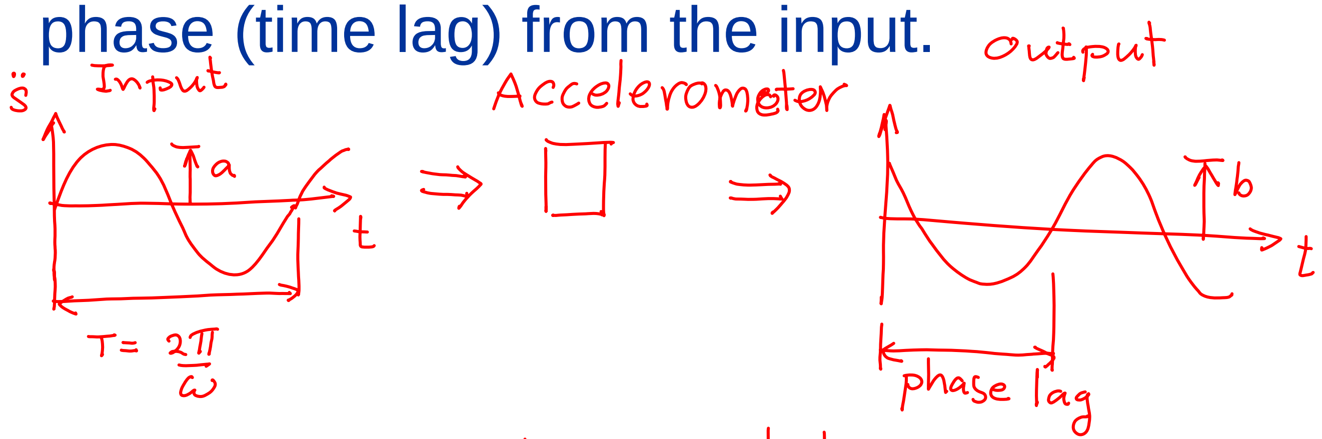
- ✓ Types of input



- ✓ Types of transducer (instrument)
- ✓ Initial conditions
- ✓ System characteristics

Introduction

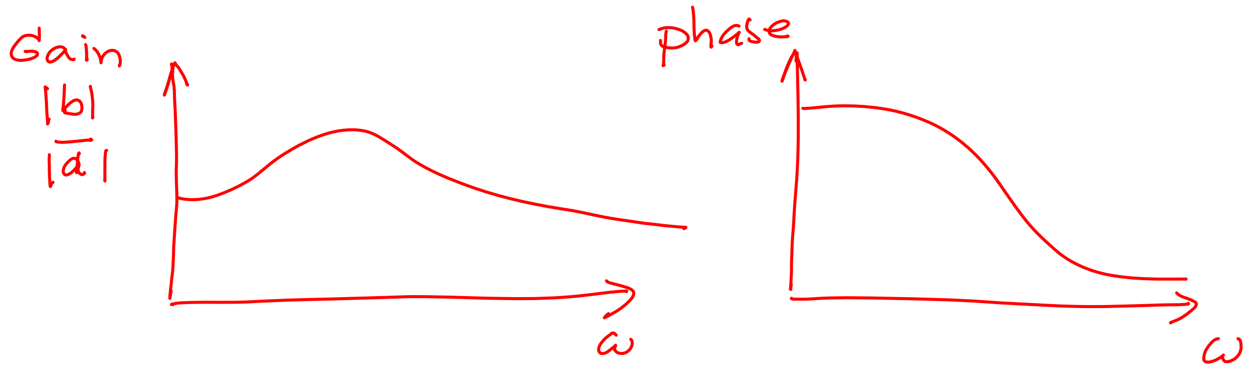
When subjected to dynamic input, e.g. sinusoidal, an instrument has a response with differences in both amplitude and phase (time lag) from the input.



$$\text{Gain (sensitivity)} = \frac{|b|}{|a|}$$

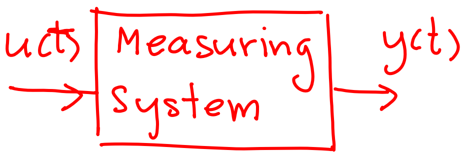
Introduction

The differences in both amplitude and phase are normally shown in Frequency Response plot.



Dynamics of Measuring Systems

- General dynamic model of a system is governed by a linear ODE:

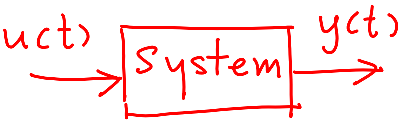


$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \dots + a_n y(t)$$
$$= b_m \frac{d^m}{dt^m}u(t) + \dots + b_0 u(t)$$

- Take Laplace transform (L.T.) with zero initial conditions to get Transfer Function:

$$G(s) \equiv \frac{\bar{Y}(s)}{\bar{U}(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

First-order System



$$\dot{y}(t) + By(t) = Au(t)$$

Rewrite as

$$\tau\dot{y}(t) + y(t) = c\,u(t)$$

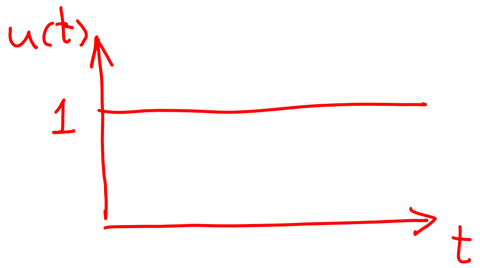
where $\tau = \frac{1}{B}$ is time constant, $c = \frac{A}{B}$

Take L.T. w/ zero initial condition $y(0) = 0$

$$\bar{Y}(s) = \frac{A}{s+B}\bar{U}(s) = \frac{c}{\tau s+1}\bar{U}(s)$$

1st-order System: Step Input

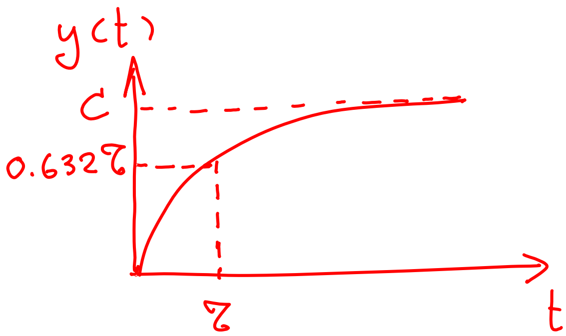
If $u(t)$ is a unit step; i.e. $u(t) = 1$



$$\bar{U}(s) = \frac{1}{s}$$

$$\therefore \bar{Y}(s) = \frac{A}{s(s+B)} = \frac{A/B}{s} - \frac{A/B}{s+B}$$

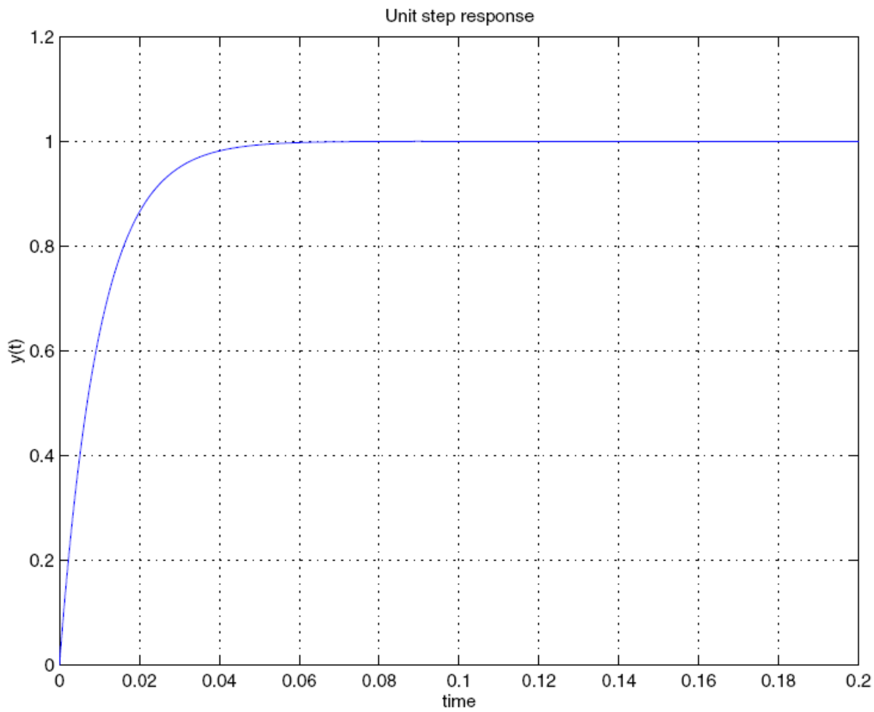
$$= \frac{c}{s} - \frac{c}{s+1/\tau}$$



Inverse L.T.

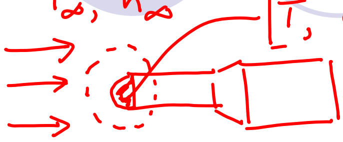
$$y(t) = c(1-e^{-Bt}) = c(1-e^{-t/\tau})$$

Plot of Unit Step Response



Ex1: Thermometer/Thermocouple

T_∞, h_∞



T, m ~ mass of sensing element (kg)

c_p ~ specific heat ($W/kg\,°c$)

T_∞ ~ surrounding fluid temp (°c)

h_∞ ~ convective heat transfer coefficient ($W/m^2\,°c$)

Convection:

$$\dot{Q} = h_\infty A (T_\infty - T) \quad \text{— (1)}$$

Conservation of energy:

$$\dot{Q} = \frac{dU}{dt} = m c_p \frac{dT}{dt} \quad \text{— (2)}$$

(1) = (2) : $h_\infty A (T_\infty - T) = m c_p \frac{dT}{dt}$

or $\tau \frac{dT}{dt} + T(t) = T_\infty$; $\tau = \frac{m c_p}{h_\infty A}$

If $T(0) = 0$, $T(t) = T_\infty (1 - e^{-t/\tau})$

what if $T(0) \neq 0$?

1st-order System: Sinusoidal Input

If $u(t) = \sin \omega t$ $\rightarrow$ $\bar{U}(s) = \dfrac{\omega}{s^2+\omega^2}$

T.F. of the 1^{st} order system is

$$G(s) = \frac{\bar{y}(s)}{\bar{U}(s)} = \frac{c}{\tau s+1}$$

$$\therefore \quad \bar{y}(s) = \frac{c}{\tau s+1}\bar{U}(s) = \frac{c\omega}{(\tau s+1)(s^2+\omega^2)}$$

Take inverse L.T.

$$y(t) = \underbrace{\frac{c\omega e^{-t/\tau}}{\tau a}}_{\text{Transient}} + \underbrace{\frac{c}{\tau a}\sin[\omega t + \phi(\omega)]}_{\text{steady state (ss)}}$$

where $a = \sqrt{\left(\frac{1}{\tau}\right)^2 + \omega^2}$

1st-order System: Sinusoidal Input

Steady state response is

$$y_{ss} = \underbrace{\frac{c}{\tau a}}_{\text{magnitude}} \sin[\omega t + \underbrace{\phi(\omega)}_{\text{phase}}]$$

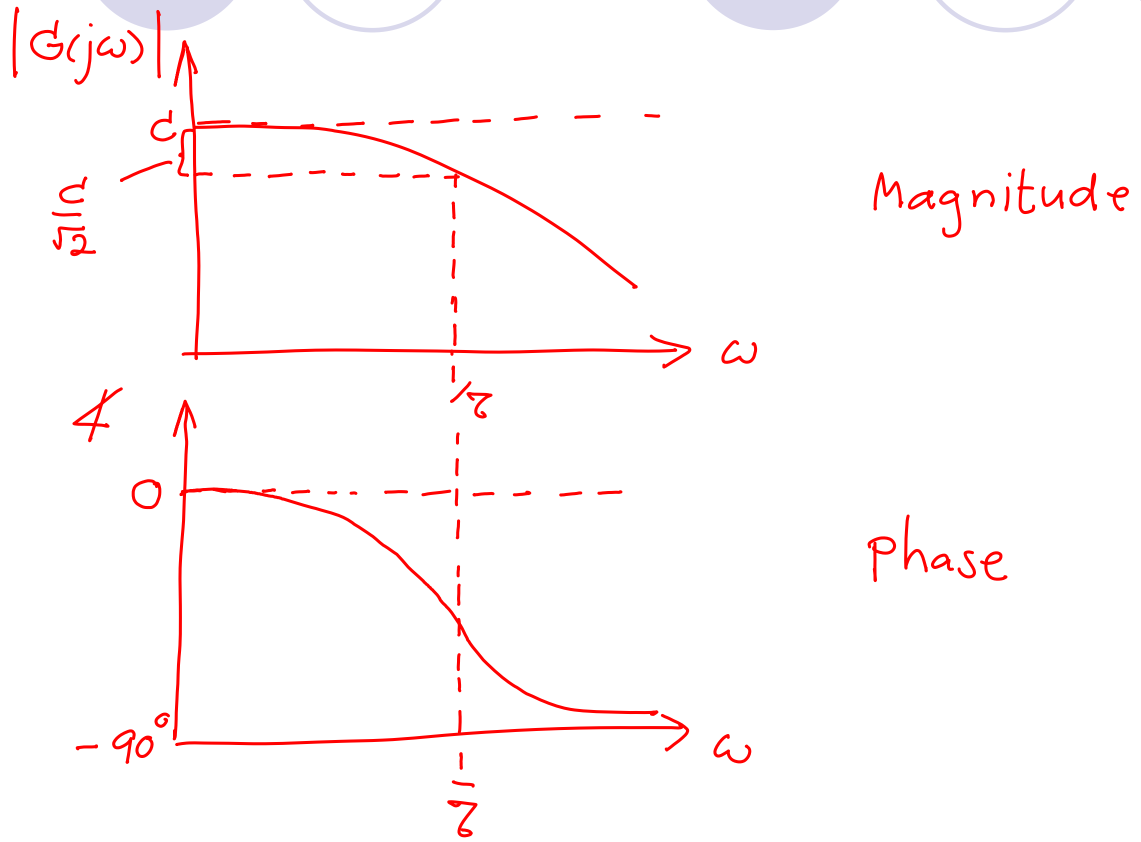
For sinusoidal response, the magnitude and phase directly relate to the transfer function $G(j\omega) = G(s)|_{s=j\omega}$ as

$$\text{Magnitude} = |G(j\omega)| = \frac{|c|}{\sqrt{(\tau\omega)^2+1}}$$

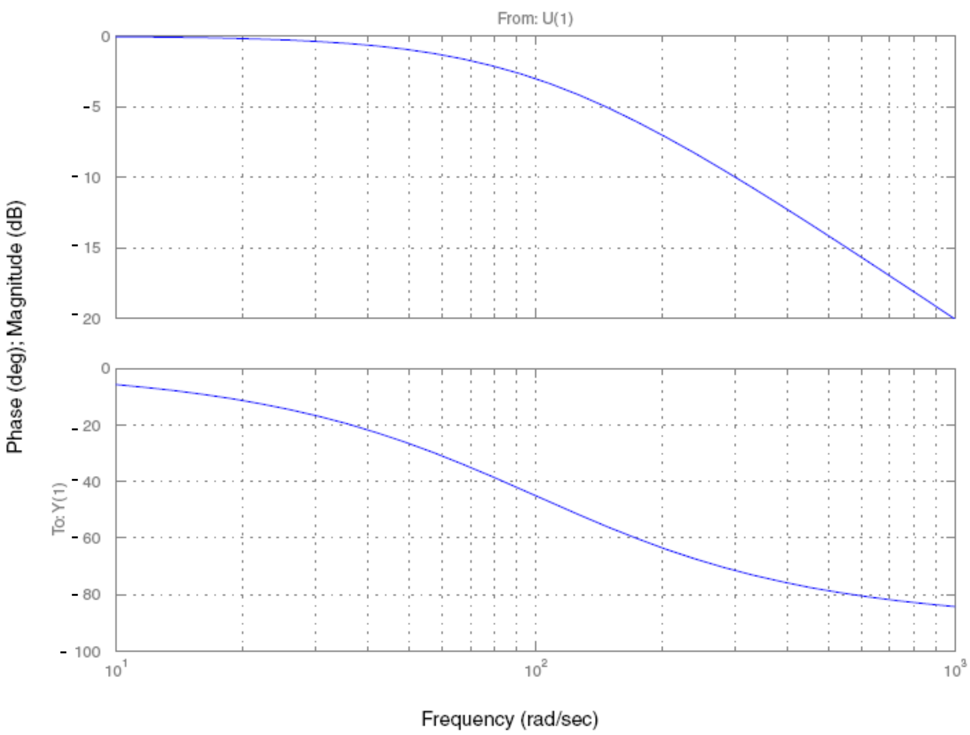
$$\phi(\omega) = \measuredangle G(j\omega) = \measuredangle\left(\frac{c}{j\omega\tau+1}\right)$$

$$= \tan^{-1}(c) - \tan^{-1}(\omega\tau)$$

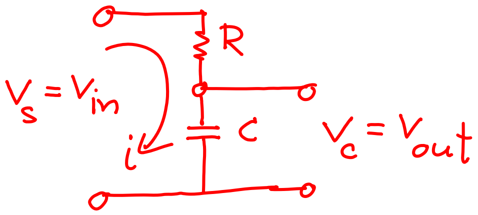
Frequency response



1st-order System: Frequency Response



Ex2: RC-Circuit (Low-Pass Filter)



$$V_s \longrightarrow \boxed{\text{system}} \longrightarrow V_c$$

Note:

R: $V_R = iR$

C: $V_c = \frac{1}{C}\int i\,dt$

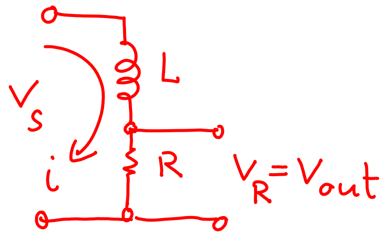
$V_L = L\frac{di}{dt}$

$$V_s = V_c + V_R$$
$$= V_c + iR$$
$$= V_c + RC\frac{dV_c}{dt} \qquad \left(\because i = C\frac{dV_c}{dt}\right)$$

or

$$\boxed{\tau\frac{dV_c}{dt} + V_c = V_s} \; ; \quad \tau = RC$$

Alternative Low-Pass Filter (RL Circuit)



$$V_s \longrightarrow \boxed{\text{system}} \longrightarrow V_R$$

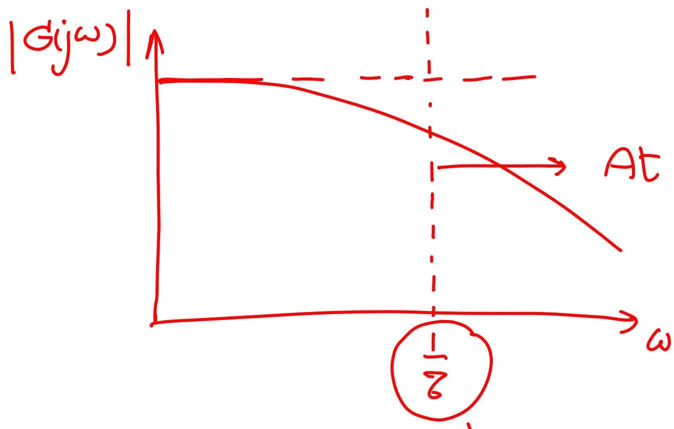
$$V_s = V_L + V_R$$
$$= L\frac{di}{dt} + V_R$$
$$= \frac{L}{R}\frac{dV_R}{dt} + V_R$$

or

$$\boxed{\tau\frac{dV_R}{dt} + V_R = V_s} \; ; \quad \tau = \frac{L}{R}$$

Low-Pass Filter: Frequency Response (Gain at Various Frequency)

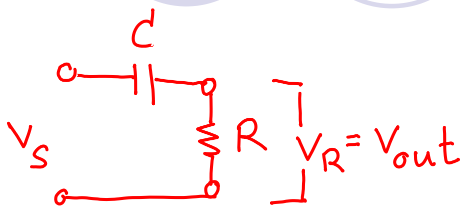
$$G(s) = \frac{1}{\tau s + 1} \rightarrow |G(j\omega)| = \frac{1}{\sqrt{(\tau\omega)^2+1}}$$

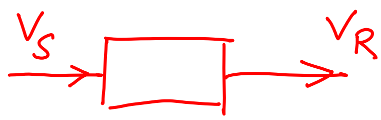


At high frequency the signal is cut-off

τ determines the cut-off frequency

Ex3: High-Pass Filter





$$V_s = V_C + V_R$$

$$= \frac{1}{C}\int i\,dt + V_R$$

$$= \frac{1}{RC}\int V_R\,dt + V_R$$

or

$$\frac{1}{RC}V_R + \frac{dV_R}{dt} = \frac{dV_s}{dt}$$

L.T.

$$\frac{1}{RC}\bar{V}_R(s) + s\bar{V}_R(s) = s\bar{V}_s(s)$$

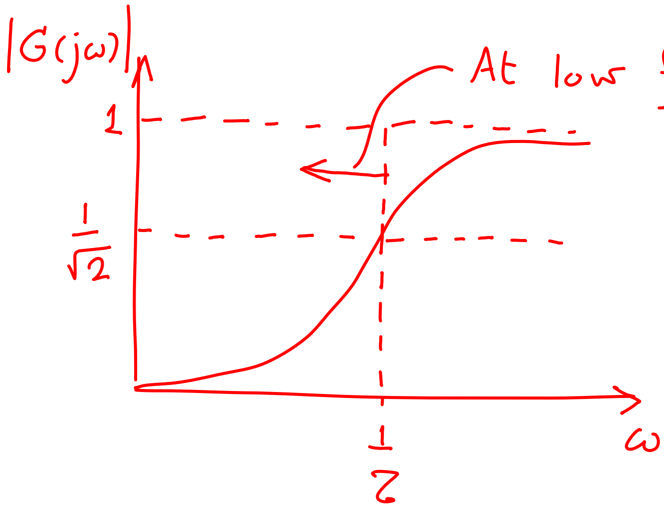
$$G(s) = \frac{\bar{V}_R(s)}{\bar{V}_s(s)} = \frac{\tau s}{1+\tau s}\ ; \quad \tau = RC$$

$$|G(j\omega)| = |G(s)|_{s=j\omega}$$

$$= \frac{\tau\omega}{\sqrt{(\tau\omega)^2+1}}$$

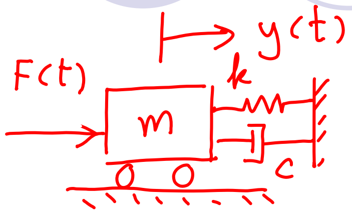
High pass filter

$$|G(j\omega)| = \frac{\tau\omega}{\sqrt{(\tau\omega)^2+1}} \quad ; \quad \tau = RC$$



At low frequency, the signal is filtered out

Second-order Systems



Accelerometer

Governing equation:

$$m\ddot{y} + c\dot{y} + ky = F(t)$$

or $$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{F(t)}{m}$$

Take L.T.

$$G(s) = \frac{\bar{Y}(s)}{\bar{F}(s)} = \frac{1}{ms^2+cs+k} = \frac{1/m}{s^2+2\zeta\omega_n s+\omega_n^2}$$

CHE roots:

$$s = -\zeta\omega_n \pm \underbrace{j\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$\zeta\omega_n$ - rate of decay
ω_d - natural freq

2nd-order System: Step Input

For $F(t) = 1$ $\longrightarrow$ $\bar{F}(s) = \frac{1}{s}$

$$\therefore y(t) = \frac{1}{k}\left[1 - e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right]$$

$\frac{1}{k}$ ~ amplitude

$\sigma = \zeta\omega_n$ ~ decay rate

ω_d ~ oscillation frequency

2nd-order System: Sinusoidal Input

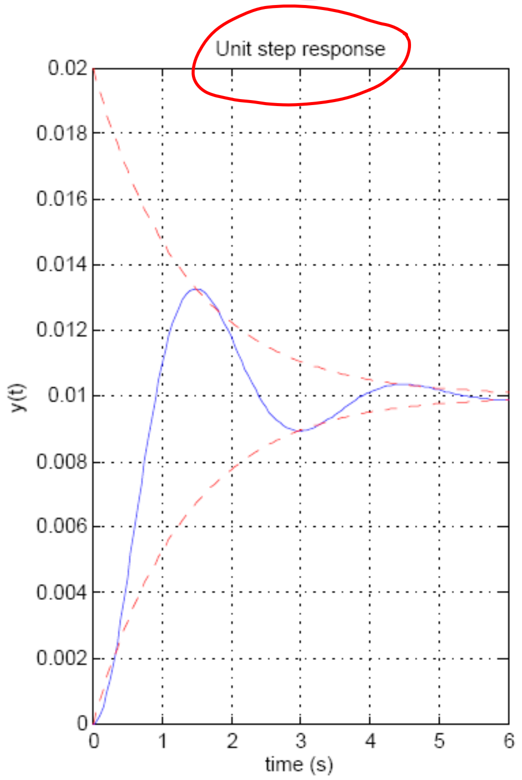
For $F(t) = \sin \omega t$

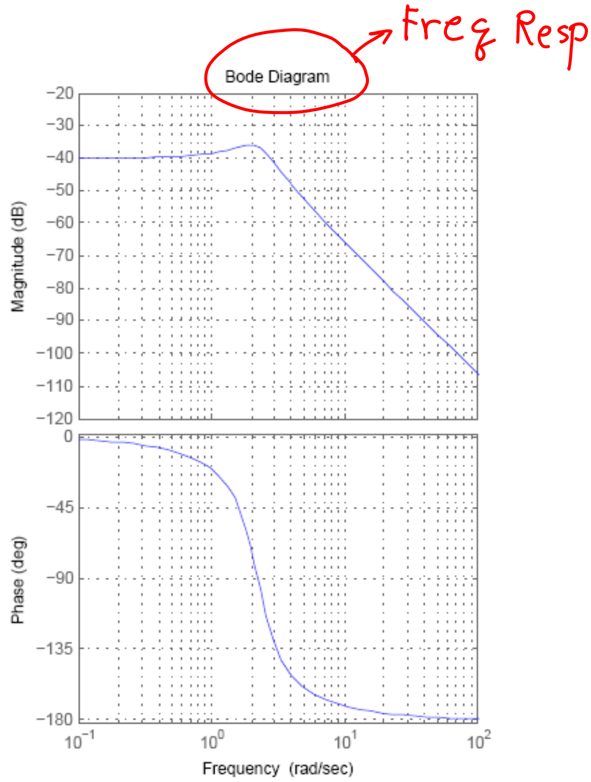
From $G(s) = \dfrac{\bar{Y}(s)}{\bar{F}(s)} = \dfrac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \dfrac{1/k}{\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta s}{\omega_n} + 1}$

$$y_{ss}(t) = |G(j\omega)| \sin[\omega t + \phi(\omega)]$$

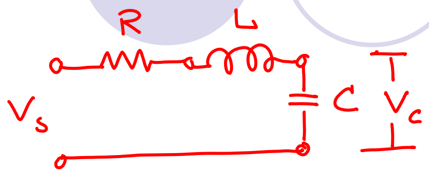
where $|G(j\omega)| = \dfrac{1/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$

and $\phi(\omega) = -\tan^{-1}\left[\dfrac{2\zeta\,\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right]$





Ex4: RLC Circuit (2^{nd} order system)



$$V_s = V_R + V_L + V_c$$

$$= iR + L\frac{di}{dt} + V_c$$

$$= CR\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c$$

$$\left(i = C\frac{dV_c}{dt}\right)$$

Or $\ddot{V}_c + 2\zeta\omega_n\dot{V}_c + \omega_n^2 V_c = V_s$

where $\omega_n = \sqrt{\frac{1}{LC}}$, $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$

Summary

- Frequency response gives information of instrument sensitivity at various frequencies; i.e., how much the output amplitude is amplified from the input, and how the output phase is changed for each frequency.

$$\underset{\sin\omega t}{u(t)} \longrightarrow \boxed{G(j\omega)} \longrightarrow y(t) \qquad G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)}$$

- For total sensitivity, all block diagram is simply combined.

$$u_I \longrightarrow \underset{\text{Inst 1}}{\boxed{G_1}} \xrightarrow{u_2} \underset{\text{Inst 2}}{\boxed{G_2}} \longrightarrow \underset{\text{Inst 3}}{\boxed{G_3}} \longrightarrow u_f$$

$$G_T = G_1 G_2 G_3 = \frac{\bar{U}_f(s)}{\bar{U}_I(s)}$$