Dynamic Characteristics of Instruments and Measuring Systems

2103-602 Measurement and Instrumentation

Introduction

Instruments and measuring system do not respond to the change of input instantly.

depend on dynamic characteristic
Output response depends on:

- Types of input

- Types of transducer (instrument)

- Initial conditions

- System characteristics

When subjected to dynamic input, e.g. sinusoidal, an instrument has a response with differences in both amplitude and phase (time lag) from the input.

\[
\text{Gain} = \frac{|b|}{|a|} \quad \text{(sensitivity)}
\]
Introduction

The differences in both amplitude and phase are normally shown in Frequency Response plot.

Dynamics of Measuring Systems

- General dynamic model of a system is governed by a linear ODE:

\[
\frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_1 y(t) = b_m \frac{d^m}{dt^m} u(t) + \ldots + b_0 u(t)
\]

- Take Laplace transform (L.T.) with zero initial conditions to get Transfer Function:

\[
G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_0} = \frac{k(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}
\]
First-order System

If $u(t)$ is a unit step; i.e. $u(t) = 1$

Rewrite as

$2y(t) + y(t) = cu(t)$

where $\tau = \frac{1}{B}$ is time constant, $c = \frac{A}{B}$

Take L.T. w/ zero initial condition $y(0) = 0$

$\tilde{y}(s) = \frac{A}{s+B} \tilde{u}(s) = \frac{C}{\tau s+1}$

$\tilde{u}(s) = \frac{1}{s}$

$\tilde{y}(s) = \frac{A}{s(s+B)} = \frac{A/B}{s} - \frac{A}{s+B}$

$= \frac{C}{s} - \frac{C}{s+\frac{1}{\tau}}$

Inverse L.T.

$y(t) = C(1-e^{-Bt}) = C(1-e^{-\frac{t}{\tau}})$
Ex1: Thermometer/Thermocouple

\[ T_s, h_{\infty}, T, c_p \text{~specific heat} \quad (\text{W/kg}^\circ\text{C}) \]

\[ T_{\infty} \text{~surrounding fluid temp (}^\circ\text{C}) \]

\[ h_{\infty} \text{~convective heat transfer coefficient} \quad (\text{W/m}^2\circ\text{C}) \]

\[ \dot{Q} = h_{\infty} A (T_{\infty} - T) \quad - (1) \]

\[ \text{Conservation of energy:} \]

\[ \dot{Q} = \frac{dT}{dt} = m c_p \frac{dT}{dt} \quad - (2) \]

\[ (1) = (2) : \quad h_{\infty} A (T_{\infty} - T) = m c_p \frac{dT}{dt} \]

or \[ \frac{dT}{dt} + T(t) = T_{\infty} ; \quad T = \frac{m c_p}{h_{\infty} A} \]

If \[ T(0) = 0, \quad T(t) = T_{\infty} (1 - e^{-t/\tau}) \]

\[ \text{what if } T(0) \neq 0 ? \]
1st-order System: Sinusoidal Input

If \( u(t) = \sin \omega t \rightarrow \bar{U}(s) = \frac{\omega}{s^2 + \omega^2} \)

T.F. of the 1st-order system is

\[
G(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \frac{C}{8s + 1}
\]

\[
\therefore \bar{Y}(s) = \frac{C}{8s + 1} \bar{U}(s) = \frac{c\omega}{(8s + 1)(s^2 + \omega^2)}
\]

Take inverse L.T.

\[
y(t) = \frac{c\omega}{8a} e^{-\frac{t}{2}} + \frac{c}{8a} \sin[\omega t + \phi(\omega)]
\]

Transient

\[
\text{where } \alpha = \sqrt{\left(\frac{1}{8}\right)^2 + \omega^2}
\]

1st-order System: Sinusoidal Input

Steady state response is

\[
y_{ss} = \frac{C}{8a} \sin[\omega t + \phi(\omega)]
\]

\[
\text{magnitude } \rightarrow \text{phase}
\]

For sinusoidal response, the magnitude and phase directly relate to the transfer function \( G(j\omega) = G(s) \mid _{s = j\omega} \) as

\[
\text{Magnitude } = |G(j\omega)| = \frac{|c|}{\sqrt{(8\omega)^2 + 1}}
\]

\[
\phi(\omega) = \angle G(j\omega) = \angle \left( \frac{c}{(8\omega)^2 + 1} \right)
\]

\[
= \tan^{-1}(c) - \tan^{-1}(\omega 2)
\]
1st-order System: Frequency Response
Ex2: RC-Circuit (Low-Pass Filter)

Note:
\[ V_R = iR \]
\[ V_C = \frac{1}{C} \int i \, dt \]
\[ V_L = L \frac{di}{dt} \]

\[ V_s = V_C + V_R = V_C + iR \]
\[ = V_C + RC \frac{dV_C}{dt} \quad (\because i = CD) \]

or

\[ L \frac{dV_C}{dt} + V_C = V_s \]
\[ \tau = RC \]

Alternative Low-Pass Filter (RL Circuit)

\[ V_s = V_L + V_R \]
\[ = L \frac{di}{dt} + V_R \]
\[ = \frac{L}{R} \frac{dV_R}{dt} + V_R \]

or

\[ L \frac{dV_R}{dt} + V_R = V_s \]
\[ \tau = LR \]
Low-Pass Filter: Frequency Response
(Gain at Various Frequency)

\[ G(s) = \frac{1}{\frac{s}{R} + 1} \rightarrow |G(j\omega)| = \frac{1}{\sqrt{(\omega R)^2 + 1}} \]

At high frequency, the signal is cut-off

\[ \frac{1}{\omega R} \] determines the cut-off frequency

Ex3: High-Pass Filter

\[ V_S = V_C + V_R \]
\[ = \frac{1}{C} \int i \, dt + V_R \]
\[ = \frac{1}{RC} \int V_R \, dt + V_R \]
\[ \text{or} \]
\[ \frac{1}{RC} V_R + \frac{dV_R}{dt} = \frac{dV_S}{dt} \]

L.T.
\[ \frac{1}{RC} \overline{V}_R(s) + s\overline{V}_R(s) = s\overline{V}_S(s) \]
\[ G(s) = \overline{V}_R(s) = \frac{s\overline{V}_S(s)}{1 + \frac{RC}{s}} \]
\[ |G(j\omega)| = |G(s)|_{s=j\omega} = \frac{\omega}{\sqrt{(\omega^2 R^2 + 1)}} \]
High pass filter

\[ |G(j\omega)| = \frac{2\omega}{\sqrt{(2\omega)^2 + 1}} \]

At low frequency, the signal is filtered out

Second-order Systems

\[ F(t) \rightarrow y(t) \]

Accelerometer

Governing equation:
\[ m \ddot{y} + c \dot{y} + ky = F(t) \]

or \[ \ddot{y} + 2\xi \omega_n \dot{y} + \omega_n^2 y = \frac{F(t)}{m} \]

Take L.T.
\[ G(s) = \frac{\bar{y}(s)}{\bar{F}(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{m} \frac{s^2 + 2\xi \omega_n s + \omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

CHE roots:
\[ s = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2} \]

\[ \omega_d \text{ - natural freq} \]

\[ \xi \omega_n \text{ - rate of decay} \]
2nd-order System: Step Input

For $F(t) = 1 \rightarrow \mathcal{F}(s) = \frac{1}{s}$

\[ y(t) = \frac{1}{k} \left[ 1 - e^{-\zeta \omega_n t} \sin (\omega_d t + \phi) \right] \]

$\frac{1}{k}$ ~ amplitude

$\zeta = \frac{\zeta}{\omega_n}$ ~ decay rate

$\omega_d$ ~ oscillation frequency

2nd-order System: Sinusoidal Input

For $F(t) = \sin \omega t$

From $\Theta(s) = \frac{\mathcal{F}(s)}{\mathcal{F}(s)} = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1/k}{(\frac{s}{\omega_n})^2 + \frac{2\zeta}{\omega_n} + 1}$

\[ y_{ss}(t) = |G(j\omega)| \sin [\omega t + \phi(\omega)] \]

where $|G(j\omega)| = \frac{1/k}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + (2\zeta \frac{\omega}{\omega_n})^2}}$

and $\phi(\omega) = -\tan^{-1} \left[ \frac{2\zeta \frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2} \right]$
Ex4: RLC Circuit

\[ V_S = V_R + V_L + V_C \]
\[ = (R + L \frac{di}{dt}) + V_C \]
\[ = CR \frac{dV_C}{dt} + LC \frac{d^2V_C}{dt^2} + V_C \]
\[ (i = C \frac{dV_c}{dt}) \]

Or
\[ \ddot{V}_C + 2\zeta \omega_n \dot{V}_C + \omega_n^2 V_C = V_S \]

where
\[ \omega_n = \sqrt{\frac{1}{LC}} \]
\[ \zeta = \frac{R}{2 \sqrt{LC}} \]
Summary

- Frequency response gives information of instrument sensitivity at various frequencies; i.e., how much the output amplitude is amplified from the input, and how the output phase is changed for each frequency.

\[ y(t) = G(j\omega)u(t) \]

\[ G(s) = \frac{\tilde{y}(s)}{\tilde{U}(s)} \]

- For total sensitivity, all block diagram is simply combined.

\[ G_T = G_1 G_2 G_3 = \frac{\tilde{U}_f(s)}{\tilde{U}_I(s)} \]