Method of Applied Math

Lecture 10: Fourier Transformation

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Applications of Fourier Series

Applications of Fourier series:

- periodic input (t variable),
- ODE on an interval $[0, L]$ (x variable).
- solve partial differential equations (next chapter).

Periodic input. If $f(t)$ is a $2L$-periodic input, then the solution

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi t}{L} + \sin \frac{n\pi t}{L})$$

Example 1

EX. Find a particular solution of the ODE

$$\frac{dy}{dt} + y = f(t),$$

where

$$f(t) = 1 + 3\cos t + 4\sin t.$$

Example 2

EX. Find a particular solution of the ODE

$$\frac{dy}{dt} + y = f(t),$$

where

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t \leq 0, \\ 1 & \text{if } 0 < t \leq \pi. \end{cases}, \quad f(t+2\pi) = f(t).$$

Hint: the Fourier series of $f(t)$ is

$$f(t) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nt.$$

Applications of Fourier Series

Fourier series can be used to solve ODE in one spatial variable x, where $x \in [0, L]$.

If $f : [0, L] \to \mathbb{R}$ is the input, one can express the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

or the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

The solution y can be obtained by plugging the Fourier series.

Example 3

EX. Find a particular solution of the ODE

$$\frac{d^2y}{dt^2} + \frac{dy}{dx} + y = 73\sin 3x \quad (0 \le x \le \pi).$$

Example 4

EX. Find a particular solution of the ODE

$$\frac{d^2y}{dx^2} + 3y = f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \pi/2 \\ 1 & \text{if } \pi < x \le \pi. \end{cases}$$

Use the following Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - (-1)^n\right)\sin nx.$$

Fourier Integral Expansion

For a $2L$-periodic function $f(x)$, the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

can be transformed into the **complex formula**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta x}, \quad \theta = \frac{\pi}{L}$$

via Euler's identity. The complex coefficients are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\theta x}dx \quad \text{(check!)}.$$

Fourier Integral Expansion

Now assume $f : (-\infty, \infty) \to \mathbb{R}$. Let $L > 0$ and define

$$f_L(x) = f(x) \quad (-L \le x \le L), \quad f_L(x + 2L) = f_L(x).$$

Then f_L is $2L$-periodic and we have

$$\begin{aligned} f_L(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\theta x} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(y) e^{-in\theta y} dy e^{in\theta x} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \int_{-L}^{L} f(y) e^{-in\theta y} dy \right\} e^{in\theta x} (\frac{\pi}{L}). \end{aligned}$$

Fourier Integral Expansion

Setting $\omega = n\theta, d\omega = \pi/L$, we have as $L \to \infty$ that

$$\begin{aligned} f(x) &= \lim_{L\to\infty} f_L(x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(y)e^{-i\omega y}dy\right\} e^{i\omega x}d\omega. \end{aligned}$$

One can justify the above identity via Riemann summation and the convergence theorem of Fourier series.

Fourier Integral Expansion Formula

Theorem Let $f : \mathbb{R} \to \mathbb{R}$. Assume f is bounded, piecewise continuous, $f'(x^-), f'(x^+)$ exists, and $\int_{-\infty}^{\infty} |f(x)|dx < \infty$.

Then f satisfies the **Fourier integral expansion formula**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(y) e^{-i\omega y} dy \right\} e^{i\omega x} d\omega.$$

Convergence theorem. The integral converges to

$$\begin{cases} f(x_0) & f \text{ continuous at } x_0, \\ \dfrac{f(x_0^-) + f(x_0^+)}{2} & \text{otherwise.} \end{cases}$$

Fourier and Inverse Fourier Transforms

Definition. For a function $f : \mathbb{R} \to \mathbb{R}$, we define $\hat{f} : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

$\hat{f}$ is called the **Fourier transform** of f.

For a function $g : \mathbb{R} \to \mathbb{R}$ we define

$$\check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x}\, d\omega.$$

$\check{g}$ is called the **Inverse Fourier transform** of g.

Other notation.

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx,$$

and

$$\mathcal{F}^{-1}[g](x) = \check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}d\omega.$$

Using the Fourier transform and the inverse Fourier transform, the Fourier integral expansion formula can be written as

$$\mathcal{F}^{-1}[\hat{f}] = f.$$

This is also called the **Fourier inversion formula**.

Example 5

EX. Find the Fourier transform and the Fourier integral expansion formula of the function

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}.$$

Example 6

EX. Find the Fourier transform and the Fourier integral expansion formula of the function

$$f(x) = \begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Note. There is a close connection between Fourier and Laplace transform!

Example 7

EX. Use the Fourier integral expansion formula:

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} e^{i\omega x} d\omega$$

to calculate the integral

$$\int_0^{\infty} \frac{\sin \omega}{\omega} d\omega.$$

Theorem.

(1) Linearity

$$\mathcal{F}[af+bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$
$$\mathcal{F}^{-1}[aF+bG] = a\mathcal{F}^{-1}[f] + b\mathcal{F}^{-1}[G]$$

(2) Differentiation

$$\mathcal{F}[f'] = i\omega\mathcal{F}[f], \quad \mathcal{F}[f''] = (i\omega)^2\mathcal{F}[f], \quad \dots$$

(3) Scaling

$$\mathcal{F}[f(kx)] = \frac{1}{k}\hat{f}(\frac{\omega}{k}).$$

(4) Convolution

$$\mathcal{F}[f*g] = \mathcal{F}[f]\cdot\mathcal{F}[g], \quad (f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Example 8

EX. Find the Fourier transform of the function

$$f(x) = xe^{-x^2},$$

by using the formula

$$\mathcal{F}[e^{-x^2}] = \sqrt{\pi}e^{-\omega^2/4}.$$