Method of Applied Math

Lecture 11: Partial Differential Equation (PDE)

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What are partial differential equations?

A **partial differential equation** (PDE) is a differential equation involving an unknown function, let say u, which depends on **two or more variables**:

$$x, y, \dots, \quad \text{and possibly } t \text{ (time)}.$$

Thus $u = u(x, y, \dots, t)$.

In the equation, there are some **partial derivatives** of u with respect to those variables:

$$u_x, \quad u_{xx}, \quad u_y, \quad u_{xy}, \quad \dots, \quad u_t, \quad u_{tt}, \quad \dots$$

Partial Differential Equations

Three important PDEs:

- **The wave equation** (e.g. vibration of string)
- **Heat equation** (e.g. heat conduction in solid)
- **Laplace's equation** (e.g. steady-state of heat equation).

Concepts:

- **Initial condition** (prescribed initial states)
- **Boundary condition** (control along boundaries)

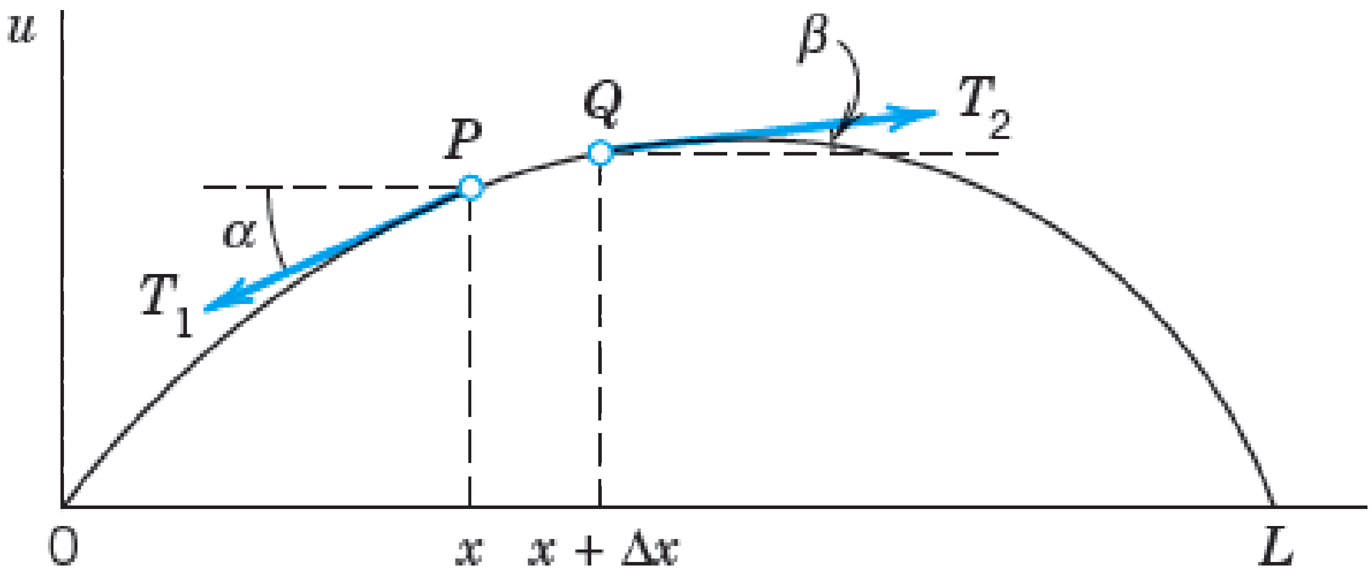
Solution techniques:

- **Fourier series**
- **Fourier transform**
- **Separation of variables**.

These techniques can be adapted to more complex problems!

We will only discuss the first two techniques.

PDE 1. Consider a rigid string moving in the vertical direction as shown in the figure below. Let $u(x,t)$ be the movement from equilibrium at a point x and time t.



The Wave Equation (Vibration Problems)

The motion of the string is the **(1-d) wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),$$

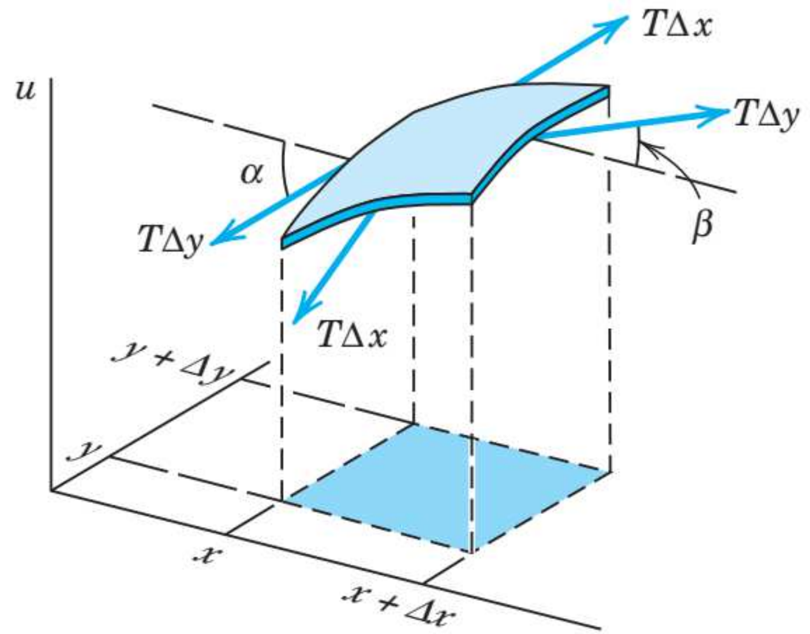
where

$$f = \text{external force},$$

and $c = \sqrt{\frac{T}{\rho}}$, $T =$ tension, $\rho =$ density.

Two-dimensional object, e.g. membrane: **(2-d) wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f(x, y, t).$$



Heat Equation (Heat Conduction Problems)

PDE 2. Consider the heat conduction on a metal bar. Let $u(x,t)$ be the temperature. The equation is **(1-d) heat equation**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x,t),$$

where

$$f = \text{source or sink}$$

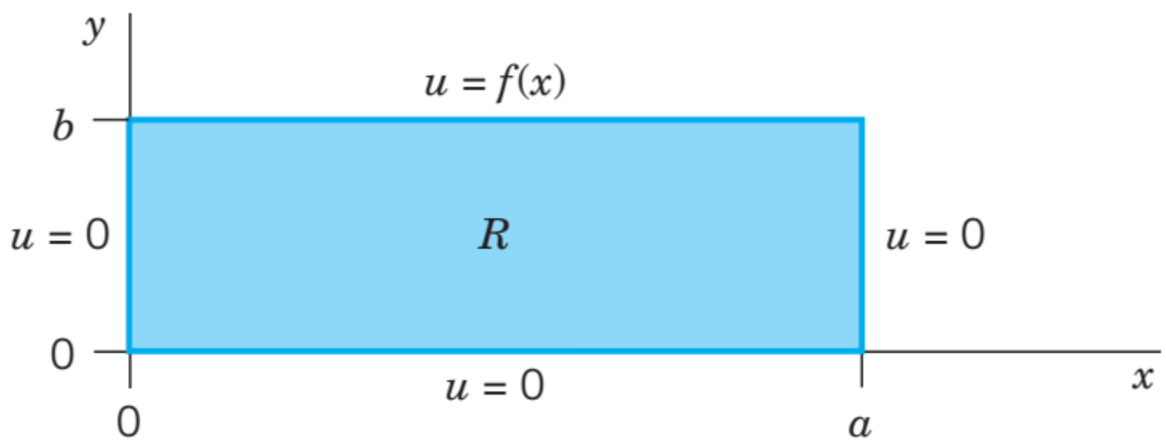
and $k > 0$ is a constant depending on the bar.

Heat Equation (Heat Conduction Problems)

Two-dimensional object, e.g. plate, **(2-d) heat equation**

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y,t), \qquad f = \text{source or sink},$$

where $u = u(x,y,t)$.



Laplace's Equation

PDE 3. The long time distribution of temperature in the heat conduction problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y)$$

for a 2-dimensional object, and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad u = u(x, y, z)$$

for a 3-dimensional object.

These are called (2-d, 3-d, respectively) **Laplace's equation**.

Some notation

- $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$
- $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \dfrac{\partial^2 u}{\partial y^2}, \quad u_{zz} = \dfrac{\partial^2 u}{\partial z^2}$
- $\Delta u = u_{xx} + u_{yy}$ if $u = u(x, y)$
- $\Delta u = u_{xx} + u_{yy} + u_{zz}$ if $u = u(x, y, z)$

Δu is called the **Laplacian of u**.

PDE: Initial Condition

If a PDE has t and the highest derivative with respect to t is n, one has to specify

n conditions

to the unknown function u at an initial time.

Heat equation $(u_t = k\Delta u + f, n = 1)$

$$u(x, 0) = \varphi(x).$$

Wave equation $(u_{tt} = c^2\Delta u + f,\ n = 2)$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$$

Laplace's equation does not need initial condition.

What are boundaries?

A bar of length L. We place $0 < x < L$. So there are 2 boundary points

- $x = 0$
- $x = L$.

A membrane or plate $W = a, L = b$. We place $0 < x < a$, $0 < y < b$. There are 4 boundary edges

- $x = 0$, $0 < y < b$
- $x = a$, $0 < y < b$
- $0 < x < a$, $y = 0$
- $0 < x < a$, $y = b$

PDE: Boundary Condition

Along each boundary, a condition must be specified.

Two standard boundary conditions: Dirichlet vs Neumann

Dirichlet type BC $(u|_{\text{boundary}} = ?)$

A bar of length L

$$u|_{x=0} = \alpha, \quad u|_{x=L} = \beta.$$

A membrane $[0,a] \times [0,b]$

$$u|_{x=0} = \alpha_1, \quad u|_{x=a} = \alpha_2$$
$$u|_{y=0} = \beta_1, \quad u|_{y=b} = \beta_2.$$

PDE: Boundary Condition

Neumann type BC $(\nabla u|_{\text{boundary}} = ?)$

A bar of length L

$$(u_x)|_{x=0} = \alpha, \quad (u_x)|_{x=L} = \beta.$$

A membrane $[0, a] \times [0, b]$

$$(u_x)|_{x=0} = \alpha_1, \quad (u_x)|_{x=a} = \alpha_2$$
$$(u_y)|_{y=0} = \beta_1, \quad (u_y)|_{y=b} = \beta_2.$$

Homogeneous BC $(u|_{\text{boundary}} = 0, \ \nabla u|_{\text{boundary}} = 0)$

$$\alpha = \beta = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.$$

PDE: IBVP

Combining PDE, IC, BC $\Rightarrow$ **initial boundary value problem (IBVP)**.

The Wave IBVP (Dirichlet-homogeneous)

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t) & 0 < x < L, t > 0 \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & t > 0 \end{cases}$$

Heat equation (Neumann-homogeneous)

$$\begin{cases} u_t = k u_{xx} + f(x,t) & 0 < x < L, t > 0 \\ u(x,0) = \varphi(x) & 0 < x < L \\ u_x(0,t) = u_x(L,t) = 0 & t > 0. \end{cases}$$

PDE, BC $\Rightarrow$ boundary value problem (BVP).

Laplace's equation (Dirichlet)

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < a, 0 < y < b \\ u(0,y) = \alpha_1, \quad u(a,y) = \alpha_2 & 0 < y < b \\ u(x,0) = \beta_1, \quad u(x,b) = \beta_2 & 0 < x < a \end{cases}$$

Laplace's equation (Dirichlet & Neumann)

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < a, 0 < y < b \\ u_x(0,y) = \alpha_1, \quad u_x(a,y) = \alpha_2 & 0 < y < b \\ u(x,0) = \beta_1, \quad u(x,b) = \beta_2 & 0 < x < a \end{cases}$$

Fourier Series Solution Technique 1

A PDE that has a spatial variable

$$0 < x < L,$$

we can use the Fourier sine/cosine series expansion in x.

Homogeneous Dirichlet BC.

$$u(0,t) = u(L,t) = 0,$$

we expand **Fourier sine series**

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}, \quad f(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}.$$

Example 1

EX. Solve the IBVP for the wave equation

$$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < \pi, t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = \sin x & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 & t > 0 \end{cases}$$

ANS. $u(x,t) = \dfrac{1}{2}\sin(2t)\sin x.$

Example 1

(continue)

Example 2

EX. Solve the IBVP for heat equation

$$\begin{cases} u_t = 3u_{xx} + t\sin x & 0 < x < \pi, t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 & t > 0 \end{cases}$$

ANS. $u(x,t) = \left(\frac{1}{9}e^{-3t} + \frac{t}{3} - \frac{1}{9}\right)\sin x$

Example 2

(continue)

Example 3

EX. Solve the BVP for Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < \pi, 0 < y < \pi \\ u(0, y) = u(\pi, y) = 0 & 0 < y < \pi \\ u(x, 0) = 0, u(x, \pi) = 1 & 0 < x < \pi, \end{cases}$$

given the Fourier sine series expansion

$$1 = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad \alpha_n = \frac{2(1 - (-1)^n)}{n\pi}.$$

ANS. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} \alpha_n \frac{\sinh(ny)}{\sinh(n\pi)} \sin nx$

Example 3

Fourier Series Solution Technique 2

Neumann Homogeneous BC. If the BC is

$$u_x(0,t) = u_x(L,t) = 0$$

we expand **Fourier cosine series**

$$u(x,t) = \frac{A_0(t)}{2} + \sum_{n=1}^{\infty} A_n(t) \cos\frac{n\pi x}{L},$$

$$f(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L}.$$

Example 4

EX. Solve the IBVP for heat equation

$$\begin{cases} u_t = ku_{xx} + 3 & 0 < x < 5, t > 0 \\ u(x,0) = 1 + \cos\frac{\pi x}{5} & 0 < x < 5 \\ u_x(0,t) = u_x(5,t) = 0 & t > 0 \end{cases}$$

ANS. $u(x,t) = 3t + 1 + e^{-k(\pi/5)^2 t}\cos x$

(continue)

Example 5

EX. Solve the BVP for Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < \pi, 0 < y < \pi \\ u(0, y) = 0, \ u(\pi, y) = y & 0 < x < \pi \\ u_y(x, 0) = u_y(x, \pi) = 0 & 0 < x < \pi \end{cases}$$

given the Fourier cosine series expansion

$$y = \frac{\pi}{2} + \sum_{n=1}^{\infty} \alpha_n \cos ny, \quad \alpha_n = \frac{2((-1)^n - 1)}{n^2 \pi}.$$

ANS. $u(x, y) = \frac{x}{2} + \sum_{n=1}^{\infty} \alpha_n \frac{\sinh(nx)}{\sinh(n\pi)} \cos(ny)$

Example 5

(continue)

Fourier Transform Technique

When the PDE has a spatial variable

$$-\infty < x < \infty,$$

we solve the problem by expanding the solution into Fourier integral, e.g.

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\omega,t) e^{ix\omega} d\omega.$$

The other way around, take the Fourier transform into the problem:

$$\hat{u}(\omega,t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t) e^{-ix\omega} dx.$$

Fourier Transform Technique

There are some useful formulas and properties

$$
\begin{aligned}
\mathcal{F}[af+bg] &= a\hat{f}+b\hat{g}, \\
\mathcal{F}[u_x(x,t)] &= i\omega\hat{u}, \\
\mathcal{F}[u_{xx}(x,t)] &= (i\omega)^2\hat{u} = -\omega^2\hat{u}, \\
\mathcal{F}[u_t(x,t)] &= \frac{\partial}{\partial t}\hat{u}(\omega,t) = \hat{u}_t \\
\mathcal{F}[u_{tt}(x,t)] &= \frac{\partial^2}{\partial t^2}\hat{u}(\omega,t) = \hat{u}_{tt}, \\
\mathcal{F}[u_{yy}(x,y)] &= \frac{\partial^2}{\partial y^2}\hat{u}(\omega,y) = \hat{u}_{yy}.
\end{aligned}
$$

Fourier Transform Technique

Warning. By taking the Fourier transform into the wave equation or heat equation,

x disappears and ω becomes a new variable.

Derivatives with respect to x becomes multiplication by $i\omega$

$$u_x \Rightarrow i\omega\hat{u}, \quad u_{xx} \Rightarrow -\omega^2\hat{u}.$$

So

$$u_{tt} = c^2 u_{xx} \quad \Rightarrow \quad \hat{u}_{tt} = -c^2\omega^2\hat{u} \quad \text{ODE in } t.$$

$$u_t = k u_{xx} \quad \Rightarrow \quad \hat{u}_t = -k\omega^2\hat{u} \quad \text{ODE in } t.$$

Fourier Transform Technique

Wave. The transformed ODE

$$\hat{u}_t = -c^2\omega^2\hat{u}$$

has solution

$$\hat{u}(\omega, t) = C(\omega)\cos(c\omega t) + D(\omega)\sin(c\omega t).$$

Heat. The transformed ODE

$$\hat{u}_t = -k\omega^2\hat{u}$$

has solution

$$\hat{u}(\omega, t) = C(\omega)e^{-(k\omega^2)t}.$$

Example 6

EX. Solve the IVP for the wave equation

$$\begin{cases} u_{tt} = 9u_{xx} & -\infty < x < \infty, t > 0 \\ u(x,0) = 0,\ \ u_t(x,0) = f(x) & -\infty < x < \infty \end{cases}$$

Express the solution in terms of the Fourier transform $\hat{f}(\omega)$.

ANS. $u(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\frac{\sin(3\omega t)}{3\omega}e^{ix\omega}d\omega$

Example 6

(continue)

Example 7

EX. Solve the IVP for heat equation

$$\begin{cases} u_t = 2u_{xx} & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \end{cases}$$

Express the solution in terms of the Fourier transform $\hat{f}(\omega)$.

ANS. $u(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-2\omega^2 t}e^{ix\omega}\,d\omega$

Example 7

(continue)

Example 8

EX. Solve the BVP for Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0 & -\infty < x < \infty, 0 < y < \pi \\ u(x,0) = 0,\ u(x,\pi) = f(x) & -\infty < x < \infty \end{cases}$$

Express the solution in terms of the Fourier transform $\hat{f}(\omega)$.

ANS. $u(x,y) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega) \frac{\sinh(y\omega)}{\sinh(\pi\omega)} e^{ix\omega} d\omega$

Example 8

(continue)

Example 9

EX. Solve the following IVP

$$\begin{cases} u_t + 6u_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \end{cases}$$

Express the solution in terms of the Fourier transform $\hat{f}(\omega)$.

ANS. $u(x,t) = f(x - 6t)$

Example 9

(continue)