Method of Applied Math

Lecture 5: Laplace Transform

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Introduction

- The **Laplace Transform** is a very efficient technique for solving **Initial Value Problems** (IVPs).
- This is a transformation technique:

$$\begin{array}{c}\text{IVP}\\ \Downarrow \\ \text{Algebraic equations}\\ \Downarrow \\ \text{Solving algebraic equations}\\ \Downarrow \\ \text{Solutions to the IVPs}\end{array}$$

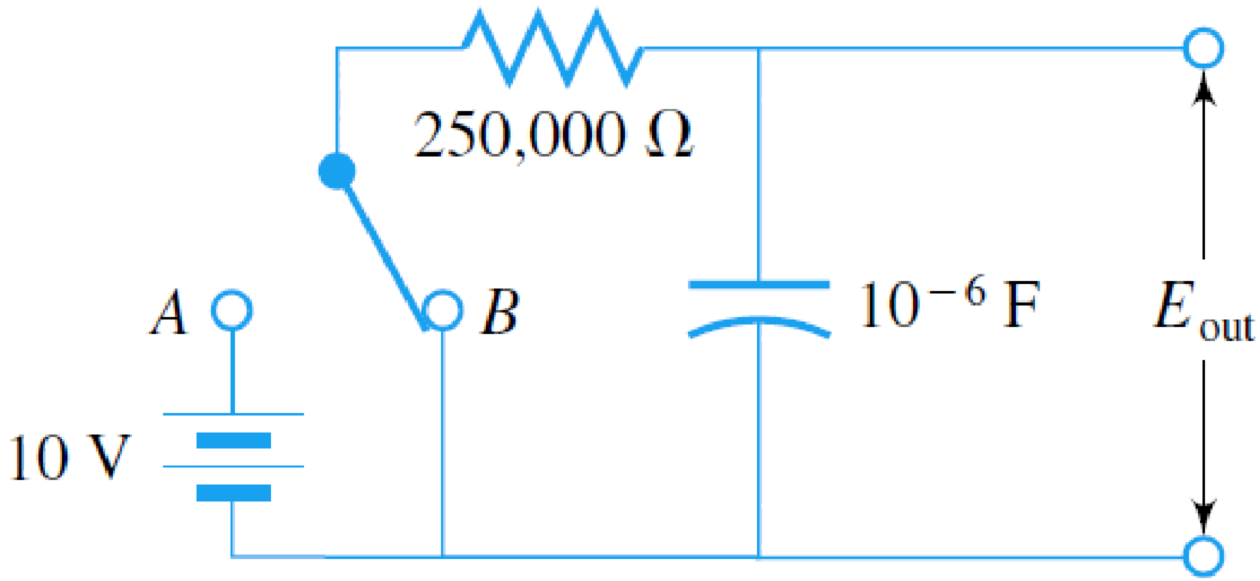
Initial Value Problems

Definition An **Initial Value Problem (IVP)** is a differential equation that is given with some conditions involving the function and some of its derivatives at a certain initial time.

- The prescribed values are called **initial conditions**.
- In practices, these values can be observed or measured by experiment.

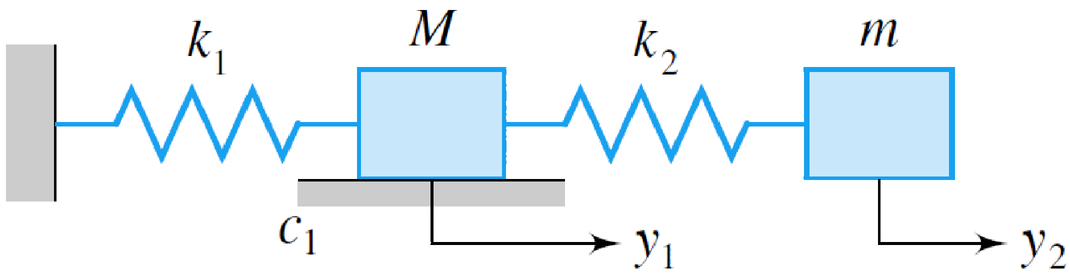
Example 1

EX. The switch is on at time $t = 2$ (from B to A), where the capacitor contained no charges before that time. After 1 min, the switch is turning back. Find the voltage drop E_{out} at any time t.



Example 2

EX. The spring-mass system is start from equilibrium at $t = 0$. Then the mass start to move at the initial velocity of 2 (m/s) to the right and is exerted by an external force $f(t) = 2\sin 5t$. Find the movement y_1 of the mass where we are given the data that $k_1 = k_2 = 2$, $M = 5$, $m = 1$, and $c_1 = 0$.



ODE of order n

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$

standard **initial conditions** are

$$y(t_0) = A_0, \quad y'(t_0) = A_1, \quad \ldots, \quad y^{(n-1)}(t_0) = A_{n-1}.$$

Usually $t_0 = 0$.

EX. The ODE

$$y'' - 3y' + 2y = e^t$$

with the initial conditions

$$y(0) = -2, \quad y'(0) = 1$$

is an initial value problem.

Initial Value Problems

EX. The ODE

$$y''' - 3y'' + y' + 5y = t \sin t$$

with the initial conditions

$$y(1) = 0, \quad y'(1) = 10, \quad y''(1) = 3$$

is an initial value problem.

Laplace Transform

Definition. Given a function $f(t)$ whose domain contains $[0, \infty)$, we define a funciton $F(s)$ by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The domain of F is any s such that the above integral converges.

$F(s)$ is called the **Lapalce transform** of $f(t)$ and is denoted by

$$F(s) = \mathcal{L}[f(t)]$$

Example 3

EX. Find the Laplace transform of

$$f(t) = e^{at}, \quad g(t) = \sin(at)$$

where a is a real number. Also, if $a > -1$ find the Laplace transform of

$$h(t) = t^a.$$

Definition. Given a function $F(s)$, a function $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$ is called **Inverse Laplace transform** of $F(s)$ and is denoted by

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

The inverse Laplace transform function $f(t)$ is unique provided it is continuous. It can be easily found from the formulas of Laplace transform.

Example 4

EX. Find the inverse Laplace transform of

$$F(t) = \frac{1}{s-a}, \quad G(t) = \frac{a}{s^2+a^2}.$$

Table

$f(t)$	$F(s)$
1	$\dfrac{1}{s} \quad (s > 0)$
$t^n \ (n = 1, 2, \ldots)$	$\dfrac{n!}{s^{n+1}} \quad (s > 0)$
$t^a \ (a > -1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}} \quad (s > 0)$
e^{at}	$\dfrac{1}{s-a} \quad (s > a)$
$\sin(at)$	$\dfrac{a}{s^2+a^2} \quad (s > 0)$
$\cos(at)$	$\dfrac{s}{s^2+a^2} \quad (s > 0)$

Linearity Property

Theorem. Let $\mathcal{L}[f(t)] = F(s), \mathcal{L}[g(t)] = G(s)$ and a, b are constants. Then

$$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$$
$$\mathcal{L}^{-1}[aF(s) + bG(s)] = af(t) + bg(t)$$

that is

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$
$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

Linearity Property

In general,

$$\mathcal{L}[af(t) + bg(t) + \cdots] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] + \cdots$$
$$\mathcal{L}^{-1}[aF(s) + bG(s) + \cdots] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)] + \cdots$$

Proof. By the linearity property for integral, we have

$$\begin{aligned}\mathcal{L}[af(t) + bg(t)] &= \int_0^\infty e^{-st}\left(af(t) + bg(t)\right)\,dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]\end{aligned}$$

Example 5

EX. Find the Laplace transform of

$$f(t) = \cos 2t + 4t + 7e^{-2t}.$$

Example 6

EX. Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s+1)(s-2)}.$$

Derivation Property

Theorem. Let $\mathcal{L}[f(t)] = F(s)$. Then

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

and

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0).$$

Generally,

$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Derivation Property

Proof. By the definition and the integration by parts, we have

$$\begin{aligned}\mathcal{L}[f'(t)] &= \int_0^\infty e^{-st} f'(t)\, dt \\ &= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt \\ &= -f(0) + sF(s)\end{aligned}$$

The remaining identity can be proved in the same way. □

Example 7

EX. Let $y(0) = -2, y'(0) = 1.$ Find the Laplace transform of

$$y'(t), \quad y''(t),$$

in terms of $Y(s) = \mathcal{L}[y(t)]$.

Example 8

EX. Assume that $y(t)$ is the solution of the IVP

$$y'' + 3y' - 2y = \sin(3t), \qquad y(0) = -1, y'(0) = 0.$$

Calculate the Laplace transform $Y(s) = \mathcal{L}[y(t)]$.

Example 9

EX. Solve the IVP

$$y' + 2y = e^{-t}, \qquad y(0) = 0$$

Example 10

EX. Solve the IVP

$$y'' + y = 3t, \qquad y(0) = 1, y'(0) = 2.$$