Method of Applied Math
Lecture 6: Laplace Transform

Sujin Khomrutai, Ph.D.

More examples

EX. Find the Laplace transform for each of the following functions $f(t)$.

1. $f(t) = 4$
2. $f(t) = 3e^{-2t}$
3. $f(t) = t^4 + t^2 + 1$
4. $f(t) = 3\sin t - 5\cos(2t)$
5. $f''(t) = e^{2t}$, $f(0) = f'(0) = 0$

More examples

EX. Find the inverse Laplace transform for each of the following functions $F(s)$.

1. $F(s) = \dfrac{1}{s-5}$
2. $F(s) = \dfrac{2}{3s+6}$
3. $F(s) = s^{-3}$
4. $F(s) = \dfrac{1}{s^2+9}$
5. $F(s) = \dfrac{s+5}{s^2+1}$
6. $F(s) = \dfrac{2}{s^2+4s+3}$

Laplace Transform of Integrals

Theorem. Let $F(s) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s} = \frac{\mathcal{L}[f(t)]}{s}$$

Also,

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(\tau)\,d\tau = \int_0^t \mathcal{L}^{-1}[F](\tau)\,d\tau$$

Laplace Transform of Integrals

Proof. Let $g(t) = \int_0^t f(\tau)d\tau$. Then $g(0) = 0$ and $g'(t) = f(t)$. Then

$$\mathcal{L}[g'(t)] = sG(s) - g(0) = sG(s).$$

On the other hand, $\mathcal{L}[g'(t)] = \mathcal{L}[f(t)] = F(s)$, hence

$$G(s) = \frac{F(s)}{s}.$$

Example 4

EX. Find the Laplace transforms

$$\mathcal{L}\left[\int_0^t e^{2\tau}\,d\tau\right], \qquad \mathcal{L}\left[\int_0^t (\tau^2 + 3\sin\tau)d\tau\right].$$

Example 5

EX. Find the inverse Laplace transforms

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right], \qquad \mathcal{L}^{-1}\left[\frac{1}{s(s-3)}\right].$$

Shifting in s

Theorem. Let $F(s) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) = \mathcal{L}[f](s-a),$$

thus

$$\mathcal{L}^{-1}\left[F(s-a)\right] = e^{at}f(t) = e^{at}\mathcal{L}^{-1}[F(s)].$$

Shifting in s

Proof. By definition,

$$\begin{aligned}
\mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\
&= \left(\int_0^\infty e^{-st}f(t)\,dt\right)\Big|_{s\to s-a} \\
&= F(s-a)
\end{aligned}$$

Example 6

EX. Find the Laplace transforms

$$\mathcal{L}\left[e^{3t}\sin t\right], \qquad \mathcal{L}[e^{-2t}\cos 5t], \qquad \mathcal{L}[t^4 e^t].$$

Example 7

EX. Find the inverse Laplace transforms

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right], \quad \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right], \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4s+5}\right].$$

Example 8

EX. Solve the IVP

$$y'' + 2y' + 5y = 0, \qquad y(0) = 0, y'(0) = 3.$$

ANS. $y(t) = \frac{3}{2}e^{-t}\sin 2t.$

Heaviside Functions

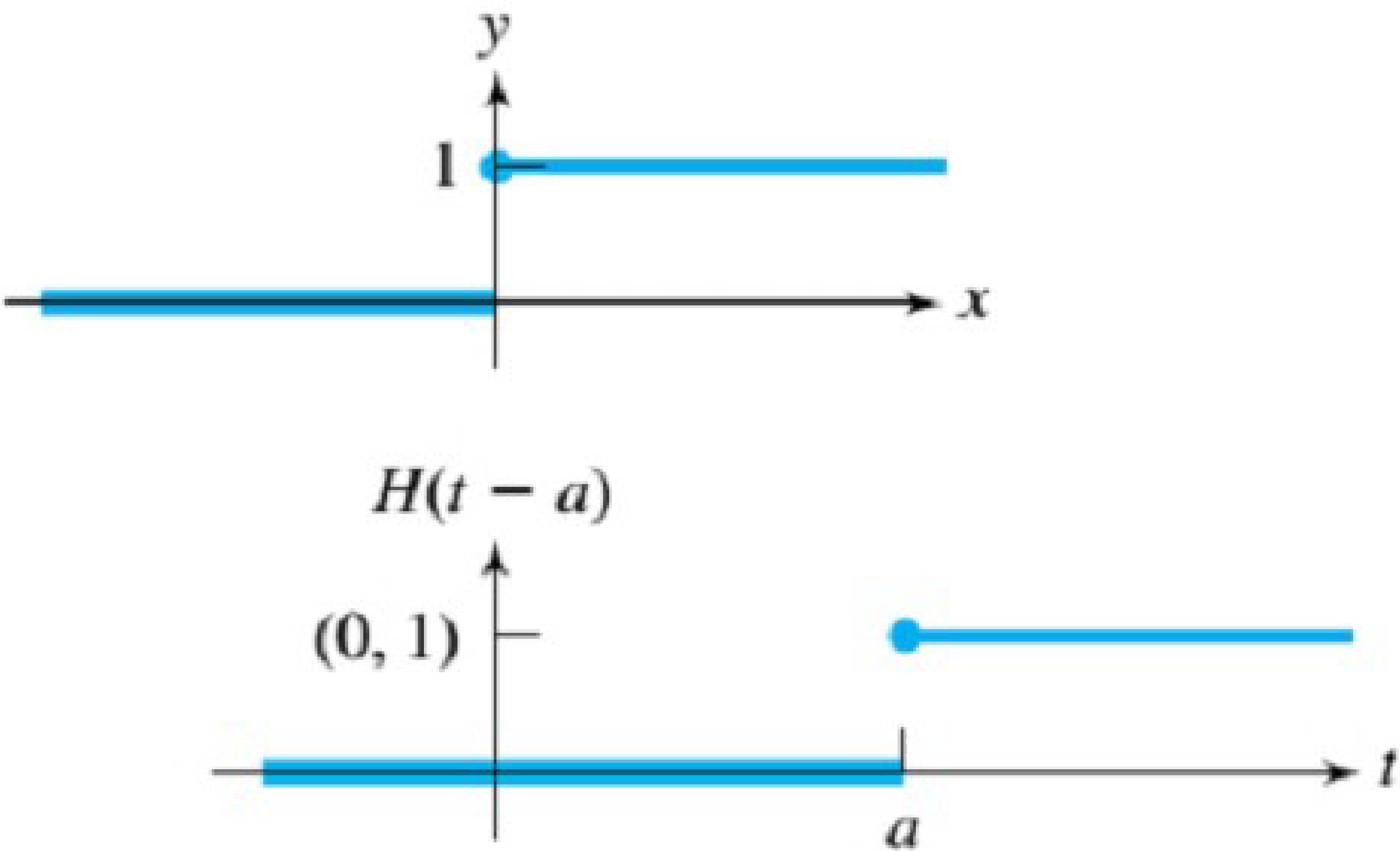
Definition. The **Heaviside function** (or **unit step function**) is

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

For a number $a \geq 0$, we have

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Heaviside Functions



The Laplace Transform of $H(t-a)$

Theorem. For a number $a \geq 0$, we have

$$\mathcal{L}[H(t-a)] = \frac{e^{-as}}{s}.$$

In particular,

$$\mathcal{L}[H(t)] = \frac{1}{s}.$$

Proof. By the definition of $H(t-a)$,

$$\mathcal{L}[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\, dt = \int_a^\infty e^{-st} dt = \frac{e^{-as}}{s}.$$

Example 9

EX. Find the Laplace transform of the following functions:

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 3 \\ 5 & \text{if } t \geq 3 \end{cases}, \quad g(t) = 3H(t) - H(t - \pi).$$

Example 9

EX. Find the inverse Laplace transforms

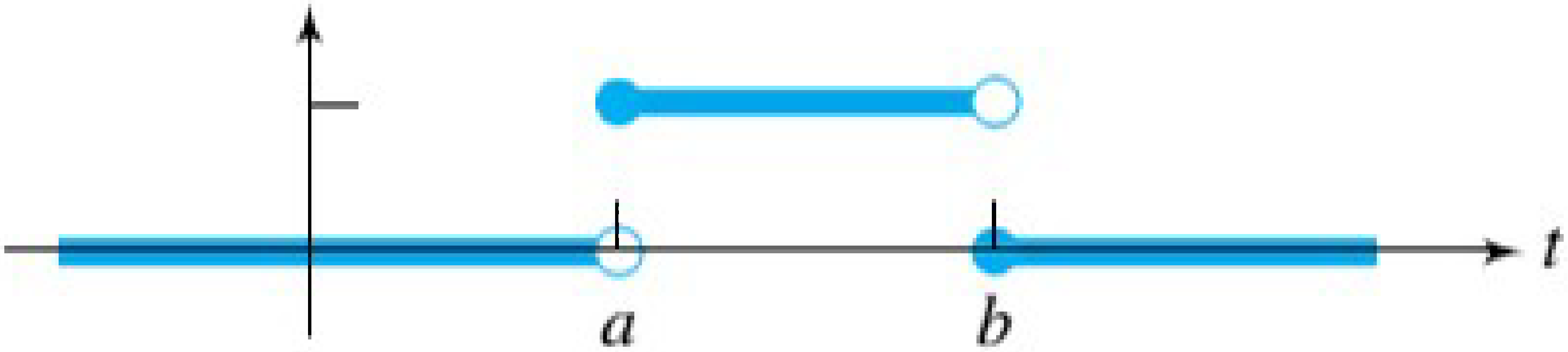
$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right], \quad \mathcal{L}^{-1}\left[\frac{2e^{-s}-e^{-2s}}{s}\right]$$

Pulses

Def. A **pulse** is a function defined by

$$k[H(t-a) - H(t-b)] = \begin{cases} 0 & \text{if } t < a \\ k & \text{if } a \leq t < b, \\ 0 & \text{if } t \geq b, \end{cases}$$

where $k \neq 0$ and $0 \leq a < b$ are constants.



Laplace Transform of Pulses

Theorem.

$$\mathcal{L}[k(H(t-a)-H(t-b))] = k\frac{e^{-as}-e^{-bs}}{s}.$$

Example 10

EX. Find the Laplace transform of the functions

$$f(t) = \begin{cases} 0 & 0 < t < 4 \\ 2 & 4 \leq t < 6 \\ 0 & t \geq 6 \end{cases}, \quad g(t) = 10(H(t-2) - H(t-5)).$$

Example 11

EX. Find the inverse Laplace transform of the function

$$F(s) = 4\frac{e^{-3s} - e^{-4s}}{s}.$$