Method of Applied Math

Lecture 7: Laplace Transform

Sujin Khomrutai, Ph.D.

Shift in t

Definition. Let a be a positive constant. The function

$$f(t-a)H(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

is called the **shifting** of $f(t)$ by a.

EX. $(t-1)^2 H(t-1)$

$$e^{t-3}H(t-3)$$

$$\sin(2t-2\pi)H(t-\pi) = \sin(2(t-\pi))H(t-\pi)$$

Shift in t

Theorem. Let $F(s) = \mathcal{L}[f(t)]$ and a a positive constant. Then

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s).$$

Thus

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)H(t-a).$$

Proof. By definition,

$$\begin{aligned}\mathcal{L}[f(t-a)H(t-a)] &= \int_0^\infty e^{-st}f(t-a)H(t-a)dt \\ &= \int_a^\infty e^{-st}f(t-a)dt.\end{aligned}$$

Shift in t

Use the change of variable $x = t - a$, the integral becomes

$$\int_a^\infty e^{-st} f(t-a)dt = \int_0^\infty e^{-s(x+a)} f(x)dx = e^{-sa}F(s).$$

So

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s).$$

Example 1

EX. Find the Laplace transform for each of the following functions

1. $f(t) = (t-3)^2 H(t-3)$

2. $g(t) = \begin{cases} 0 & 0 < t < 1 \\ e^{t-1} & t \geq 1 \end{cases}$

3. $h(t) = \begin{cases} 0 & 0 < t < 4 \\ 2t-8 & t \geq 4 \end{cases}$

Shift in t

Note. A function of the form

$$f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & t \geq a \end{cases}$$

is equal to

$$\begin{aligned} f(t) &= f_1(t)(1 - H(t-a)) + f_2(t)H(t-a) \\ &= f_1(t) - f_1(t)H(t-a) + f_2(t)H(t-a) \end{aligned}$$

EX.

$$f(t) = \begin{cases} t-5 & 0 < t < 2 \\ \cos t & t \geq 2 \end{cases} = (t-5)(1-H(t-2))+(\cos t)H(t-2)$$

Multiplication with Heaviside Function

Theorem.

$$\mathcal{L}[f(t)H(t-a)] = e^{-as}\mathcal{L}[f(t+a)].$$

Proof. Let $g(t) = f(t+a)$. Then $g(t-a) = f(t)$ hence

$$f(t)H(t-a) = g(t-a)H(t-a).$$

By the previous theorem,

$$\mathcal{L}[f(t)H(t-a)] = \mathcal{L}[g(t-a)H(t-a)] = e^{-as}G(s)$$

Since $G(s) = \mathcal{L}[g(t)] = \mathcal{L}[f(t+a)]$, the desired identity is true.

Example 2

EX. Find the Laplace transform for each of the following functions

1. $f(t) = tH(t-3)$
2. $g(t) = e^{-t}H(t-1)$
3. $h(t) = \begin{cases} 1 & 0 < t < 7 \\ \cos t & t \geq 7 \end{cases}$

Example 3

EX. Find the inverse Laplace transform for each of the following functions

1. $F(s) = \dfrac{e^{-3s}}{s-5}$

2. $G(s) = \dfrac{e^{-5s}}{s^2}$

3. $(*)\ P(s) = \dfrac{e^{-2s}}{s(s^2+1)}$

Example 4

EX. Solve the IVP:

$$y'' + 4y = f(t), \quad y(0) = 1, y'(0) = 0.$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 4 \\ 3 & \text{if } t \geq 4 \end{cases}$$

ANS.

$$y(t) = \cos 2t + \frac{3}{4}(1 - \cos(2t - 8))H(t - 4)$$

Impulse functions

Definition Let $a \geq 0$. The **unit impulse function at** a or **Dirac delta function at** a is

$$\delta(t-a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)].$$

Note. It can be proved that the following identity holds

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a),$$

for any continuous function f.

Laplace Transform of $\delta(t-a)$

Theorem For each $a \geq 0$, we have

$$\mathcal{L}\left[\delta(t-a)\right] = e^{-as}.$$

Thus

$$\mathcal{L}^{-1}\left[e^{-as}\right] = \delta(t-a).$$

In particular,

$$\mathcal{L}[\delta(t)] = 1, \quad \mathcal{L}^{-1}[1] = \delta(t).$$

Example 5

EX. Find the Laplace transform for each of the following functions

1. $f(t) = 2\delta(t)$
2. $g(t) = \delta(t-1) + 3\delta(t-5)$
3. $h(t) = 3\delta(t-4) - \delta(t-8)$

Example 6

EX. Find the inverse Laplace transform for each of the following functions

1. $F(s) = 7$
2. $G(s) = 3e^{-4s}$
3. $P(s) = 2 - e^{-s} + 3e^{-2s}$

Example 7

EX. Solve the IVP

$$y'' + 9y = \delta(t-1), \qquad y(0) = y'(0) = 0$$

ANS.

$$y(t) = \frac{1}{3}\sin(3t-3)H(t-1)$$

Question 1

Q. Find the Laplace transform of

1. $f(t) = t^2 H(t-3)$
2. $g(t) = 2\delta(t) - 5\delta(t-9)$

Question 2

Q. Find the inverse Laplace transform of

1. $$F(s) = \frac{e^{-9s}}{s^2+1}$$
2. $$G(s) = \frac{e^{2s}+5}{e^{3s}}$$

Convolution

Q. Given two Laplace transforms

$$F(s) = \mathcal{L}[f(t)], \quad G(s) = \mathcal{L}[g(t)],$$

does there exists a function/formula having the Laplace transform

$$F(s)G(s)?$$

Definition. The **convolution** of two function $f(t), g(t)$ is the function, denoted $f * g$, given by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Example 8

EX. Find the following convolutions

1. $1 * 1$
2. $1 * e^t$
3. $t * t$

Convolution

Theorem.

$$\mathcal{L}[(f*g)(t)] = F(s)G(s)$$

so

$$\mathcal{L}^{-1}[F(s)G(s)] = (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Proof. We have

$$\mathcal{L}[(f*g)(t)] = \int_0^\infty e^{-st}\int_0^t f(t-\tau)g(\tau)d\tau dt = F(s)G(s).$$

Example 9

EX. Find the Laplace transform for each of the following functions

1. $f(t) = (t * \sin t)$

2. $g(t) = \displaystyle\int_0^t (t-\tau)e^{\tau} d\tau$

3. $h(t) = \displaystyle\int_0^t e^{t-\tau} \cos 2\tau d\tau$

Example 10

EX. For each of the following functions, express the inverse Laplace transform in terms of the convolution

1. $F(s) = \dfrac{s}{(s^2+4)(s^2+1)}$

2. $G(s) = \dfrac{1}{s(s^2+1)}$

3. $P(s) = \dfrac{1}{s^4(s-5)}$

Example 11

EX. Using the convolution, find the solution formula for each of the following IVPs

1. $y'' - 5y' + 6y = f(t), \quad y(0) = y'(0) = 0$
2. $y'' + 10y' + 24y = f(t), \quad y(0) = 1, y'(0) = 0$
3. $y'' - 4y' - 5y = f(t), \quad y(0) = 2, y'(0) = 1$