Method of Applied Math

Lecture 8: Fourier Series

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Introduction

In this chapter we study

1. periodic functions, i.e. satisfying $f(x+P) = f(x)$
2. Fourier series representation
3. Solutions to IVP with a periodic external force.
4. functions on an interval $f : [a, b] \to \mathbb{R}$
5. Fourier sine/cosine series
6. Solutions to boundary value problems (BVPs).

EX. Forced Vibration

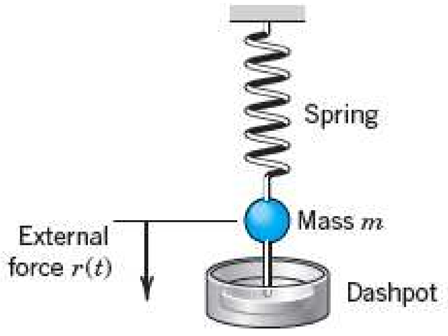


Fig. 274. Vibrating system under consideration

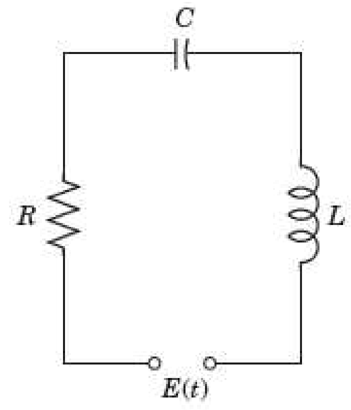


Fig. 275. Electrical analog of the system in Fig. 274 (*RLC-circuit*)

Mathematical model:

$$my'' + \gamma y' + ky = r(t)$$

Example

$r(t)$ is periodic

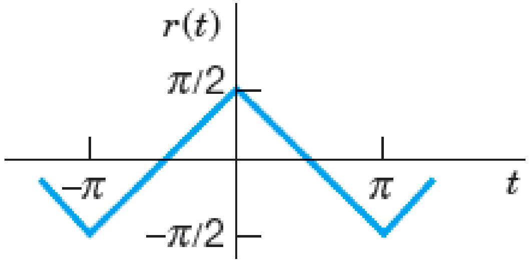


Fig. 276. Force in Example 1

For such $r(t)$, we will express $r(t)$ as a **Fourier series:**

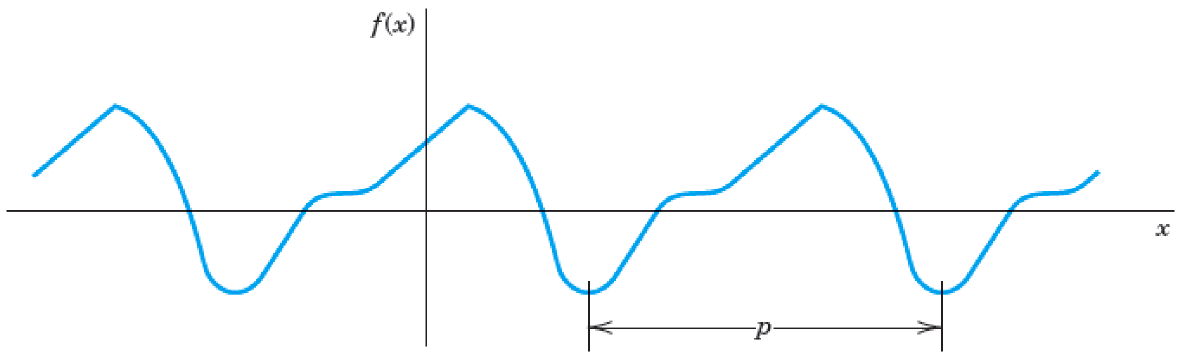
$$r(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Periodic Functions

Definition For a function $f(x)$, if there is $p > 0$ such that

$$f(x+p) = f(x) \qquad \text{for all } x,$$

$f(x)$ is called a **periodic function** and p is a **period** of $f(x)$, or simply f is p-**periodic**.

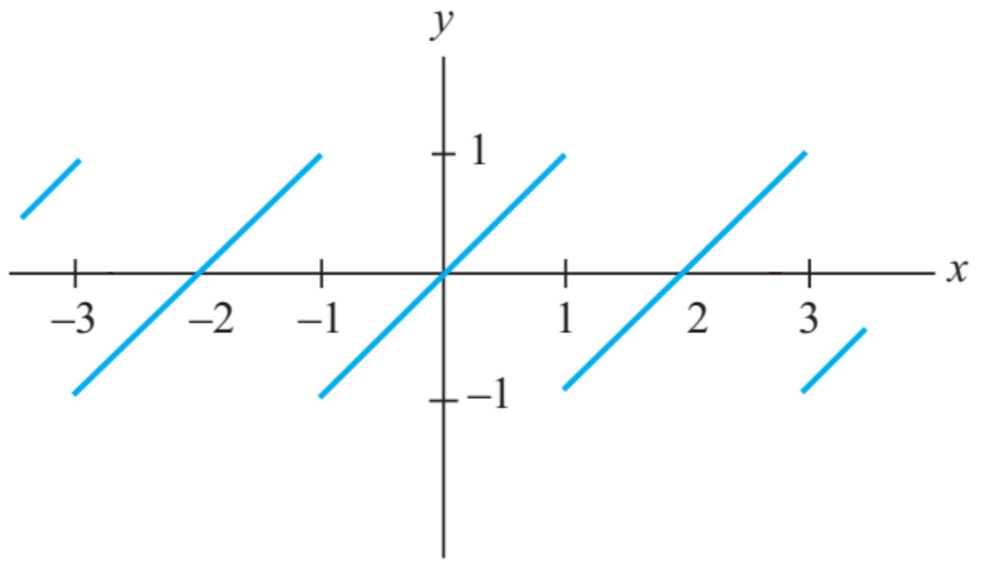


Example 1

EX. The function f defined by

$$f(x) = x \quad \text{for } -1 < x < 1, \qquad f(x+2) = f(x) \quad \text{for all } x,$$

is 2-periodic.

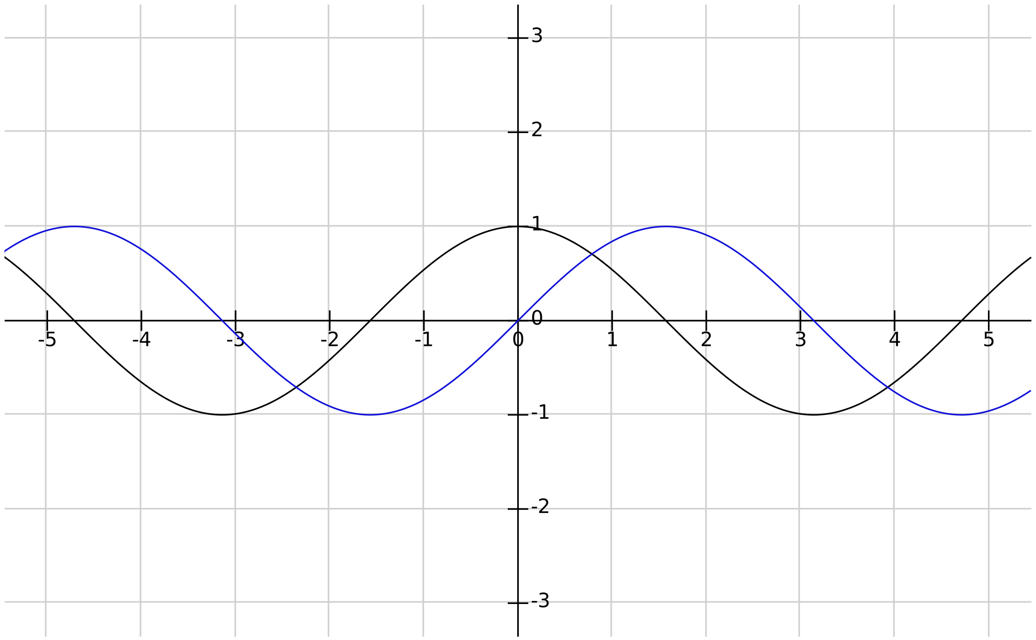


Example 2

EX. Use the well-known identity

$$\cos(A + 2\pi) = \cos A, \quad \sin(A + 2\pi) = \sin A$$

so show that $f(x) = \cos x$ and $g(x) = \sin x$ are 2π-periodic.

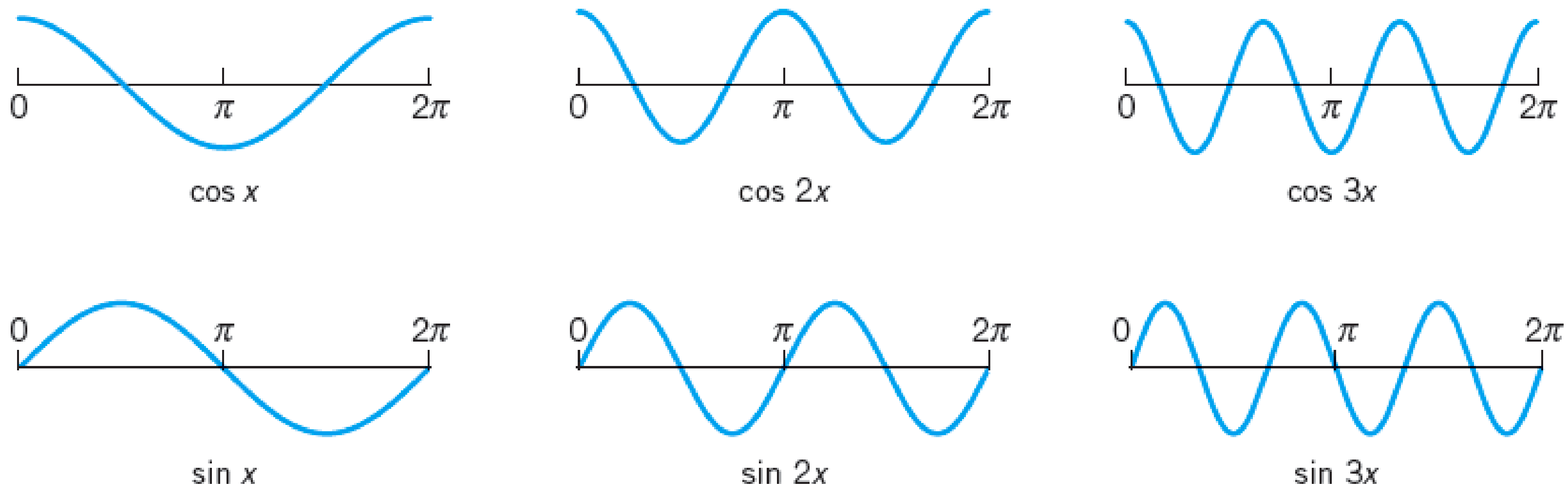


Example 3

EX. Generally, for $n = 1, 2, \ldots$, show that the functions

$$f_n(x) = \cos nx \quad \text{and} \quad g_n(x) = \sin nx$$

are 2π-periodic.



Example 4

EX. Let $L > 0$. Show that

$$f(x) = \cos\frac{\pi x}{L} \quad \text{and} \quad g(x) = \sin\frac{\pi x}{L}$$

are $2L$-periodic. Prove that the same conclusion holds for

$$f_n(x) = \cos\frac{n\pi x}{L} \quad \text{and} \quad g_n(x) = \sin\frac{n\pi x}{L},$$

for $n = 1, 2, \ldots$

Question 3

Q. Show that

$$f(x) = \sin^2 3x$$

is a 2π-periodic function.

Note. If f, g are P-periodic functions, then so is $af + bg$, for any constants a, b. In fact,

$$\begin{aligned}(af + bg)(x + P) &= af(x + P) + bg(x + P) = af(x) + bg(x) \\ &= (af + bg)(x).\end{aligned}$$

This fact is applied to

$$a_1 f_1 + a_2 f_2 + \cdots$$

Plan

1. $P = 2\pi$
2. $P = 2L$, $L > 0$ arbitrary.

Fourier Series: Period 2π

Definition Given numbers a_0, a_n, b_n $(n = 1, 2, \ldots)$, a series of the form

$$\frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

is called a **Fourier series with period** 2π (the right hand side is 2π-periodic).
a_0, a_n, b_n's are called **coefficients**.

Fourier Series: Period 2π

Theorem Let $f(x)$ be a 2π-periodic function. Assume that

1. f is piecewise continuous, and
2. $f'(x^-), f'(x^+)$ exist at every x.

Then f can be expanded into a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

(See the computation of a_0, a_n, b_n below).

This series is called the **Fourier series expansion** of $f(x)$.

Note. Piecewise continuous means there are finitely many jump discontinuities in one period.

Computation

Theorem If f is 2π-periodic with the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then the coefficients are computed from

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

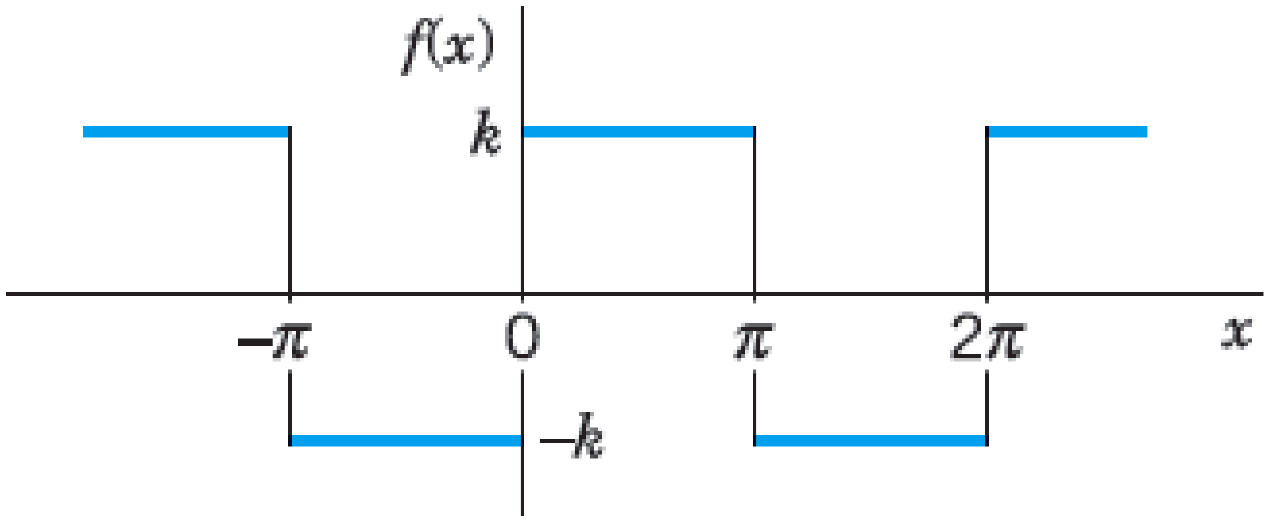
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \quad (n = 1, 2, \dots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \quad (n = 1, 2, \dots)$$

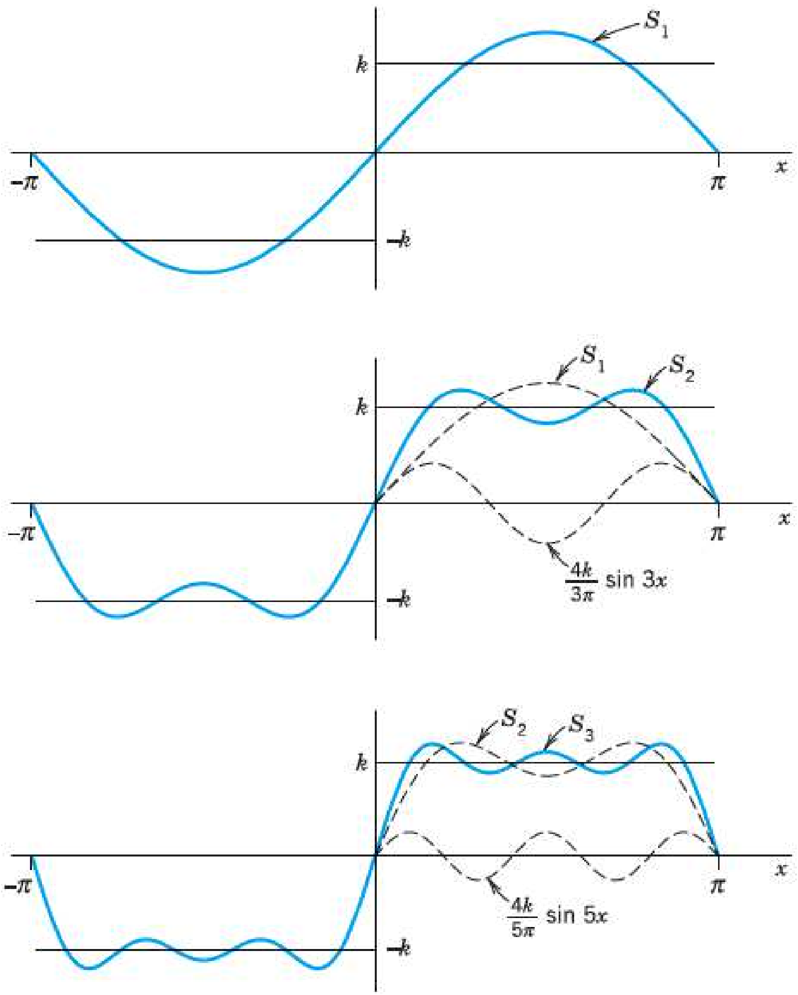
Example 5

EX. Find the Fourier series expansion of

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$



Example 5



Example 6

EX. Find the Fourier series expansion of

$$f(x) = 1 + \cos 2x.$$

ANS.

$$a_0 = 2, \quad a_1 = b_1 = 0, \quad a_2 = 1, b_2 = 0, \quad a_3 = b_3 = \cdots = 0$$

Example 7

EX. Find the Fourier series expansion of

$$f(x) = -2 + \sin x - 5\cos 2x.$$

ANS.

$$a_0 = -4, \quad a_1 = 0, \quad b_1 = 1, \quad a_2 = -5, \quad b_2 = 0,$$
$$a_3 = b_3 = a_4 = b_4 = \cdots = 0.$$

Question

Q. Use the identity

$$\cos^2 A = \frac{1+\cos 2A}{2},$$

to find the Fourier series expansion of

$$f(x) = \cos^2 x.$$

Example 8

EX. Use the integral formulas

$$\int x \cos nx dx = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + C,$$
$$\int x \sin nx dx = -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} + C,$$

to find the Fourier series expansion of a 2π-periodic function f where

$$f(x) = x \quad \text{for } -\pi < x < \pi.$$

Convergence of Fourier Series

Theorem Let $f(x)$ be a 2π-periodic function. Assume that

1. f is piecewise continuous, and
2. $f'(x^-), f'(x^+)$ exist at every x.

Then sum of the Fourier series of f satisfies

1. if $f(x)$ is continuous at x_0, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx_0) + b_n \sin(nx_0)) = f(x_0);$$

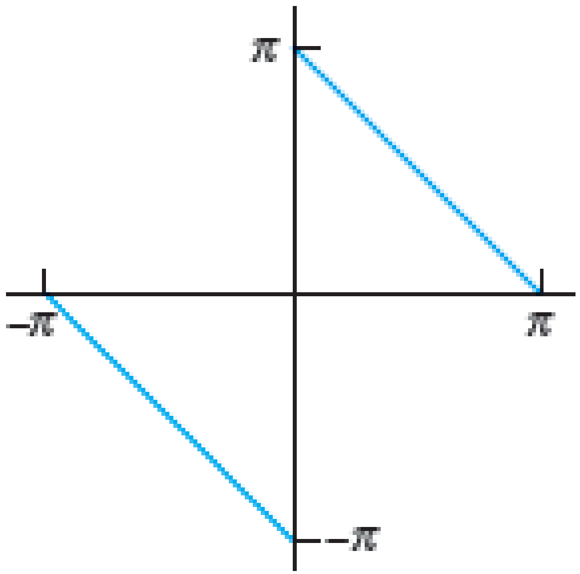
2. if f is discontinuous at x_0, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx_0) + b_n \sin(nx_0)) = \frac{f(x_0^-) + f(x_0^+)}{2}.$$

Example 9

EX. The following figure shows a 2π-periodic function f in one period. Discuss the sum of the Fourier series expansion of f at x_0 where

1. $x_0 \in (-\pi, \pi)$
2. $x_0 = -\pi$ and $x_0 = \pi$.



Example 10

EX. Given that a 2π-periodic function f where

$$f(x) = \begin{cases} -\pi/4 & \text{if } -\pi < x < 0 \\ \pi/4 & \text{if } 0 \leq x < \pi \end{cases}$$

has the Fourier series expansion

$$\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots ,$$

evaluate the sum

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$