Method of Applied Math

Lecture 9: Fourier Series

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Fourier Series: $2L$-Periodic Functions

Next we study Fourier series expansion for $2L$-periodic functions.

Typical $2L$-periodic functions are

$$\frac{1}{2}, \quad \cos\frac{\pi x}{L}, \quad \sin\frac{\pi x}{L}$$

and

$$\cos\frac{n\pi x}{L}, \quad \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Actually, these functions form a basic building block for the Fourier series of arbitrary $2L$-periodic functions.

Fourier Series: $2L$-Periodic Functions

Theorem Let $f(x)$ be a $2L$-periodic function. Assume

1. f is peicewise continuous
2. $f'(x^-), f'(x^+)$ exists for all x.

Then f has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \quad \text{where}$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \qquad (n = 1, 2, 3, \dots)$$

Example 1

EX. Find the Fourier series of the function

$$f(x) = x \quad (-2 \le x \le 2), \quad f(x+4) = f(x).$$

Note.

$$\int x \cos ax dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a} + C$$
$$\int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} + C$$
$$\cos n\pi = (-1)^n, \quad \sin n\pi = 0$$
$$\cos \frac{(2k+1)\pi}{2} = 0, \quad \sin \frac{(2k+1)\pi}{2} = (-1)^k.$$

A useful Lemma

Lemma. If a function

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{N} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right),$$

then the Fourier coefficients are

$$a_0 = A_0, \quad a_n = A_n, b_n = B_n \quad (1 \le n \le N),$$
$$a_n = b_n = 0 \quad (n \ge N+1).$$

Proof. There is a unique Fourier series for each function, so the Lemma is true by comparing coefficients.

Example 2

EX. Find the Fourier series for each of the following functions.

1. $f(x) = 3 \; (-1 \leq x \leq 1)$
2. $g(x) = \cos \pi x$
3. $h(x) = \dfrac{3}{2} + \sin \pi x - 2\cos 3\pi x$

Convergence

Theorem Let $f(x)$ be a $2L$-periodic function. Assume as the preceding theorem.

1. If f is continuous at x_0, then the sum of the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x_0}{L} + b_n \sin \frac{n\pi x_0}{L} \right) = f(x_0).$$

2. If f is discontinuous at x_0, then the sum of the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x_0}{L} + b_n \sin \frac{n\pi x_0}{L} \right) = \frac{f(x_0^-) + f(x_0^+)}{2}.$$

Example 3

EX. Find the sum of the Fourier series for the function

$$f(x) = \begin{cases} 0 & -2 < x \leq -1 \\ 5 & -1 < x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}, \quad f(x+4) = f(x).$$

Example 4

EX. Find the sum of the Fourier series for the function

$$f(x) = \begin{cases} -3 & -1 < x \leq 0 \\ 3 & 0 < x \leq 1 \end{cases}, \quad f(x+2) = f(x).$$

Even Functions, Odd Functions

Definition A function $f(x)$ is called an **even function** if

$$f(-x) = f(x) \qquad \forall\, x.$$

$f(x)$ is called an **odd function** if

$$f(-x) = -f(x) \qquad \forall\, x.$$

- $f_1(x) = x^2$, $f_2(x) = a_0/2$, $f_3(x) = \cos \dfrac{n\pi x}{L}$ are even functions.
- $g_1(x) = x$, $g_2(x) = \sin \dfrac{n\pi x}{L}$ are odd functions.

Fourier Series: Even & Odd Periodic Functions

Theorem Let $f(x)$ be a $2L$-periodic function.

- If f is **even**, then $b_n = 0$. So the Fourier series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,$$

for $n = 0, 1, 2 \ldots$

- If f is **odd**, then $a_n = 0$. So the Fourier series of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

for $n = 1, 2, \ldots$

EX. Find the Fourier series of the function

$$f(x) = |x| \quad (-\pi \leq x \leq \pi), \quad f(x+2\pi) = f(x).$$

Note.

$$\int x \cos ax dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a} + C$$

$$\int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} + C$$

$$\cos n\pi = (-1)^n, \quad \sin n\pi = 0$$

$$\cos \frac{(2k+1)\pi}{2} = 0, \quad \sin \frac{(2k+1)\pi}{2} = (-1)^k.$$

Fourier Series: Linear Combination

Note. If f, g are $2L$-periodic with Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$g(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos \frac{n\pi x}{L} + d_n \sin \frac{n\pi x}{L} \right)$$

and k, l are constants, then $kf + lg$ has the Fourier series

$$\frac{a_0 + c_0}{2} + \sum_{n=1}^{\infty} \left((a_n + c_n) \cos \frac{n\pi x}{L} + (b_n + d_n) \sin \frac{n\pi x}{L} \right)$$

Example 6

EX. Find the Fourier series expansion of

$$f(x) = x + \pi \quad (-\pi < x < \pi), \quad f(x + 2\pi) = f(x).$$

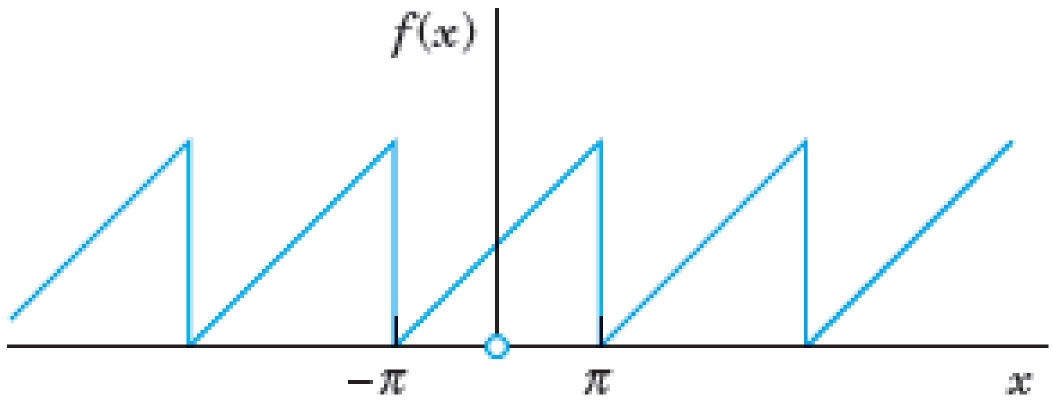


Fig. 268. The function $f(x)$. Sawtooth wave

Example 6

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right).$$

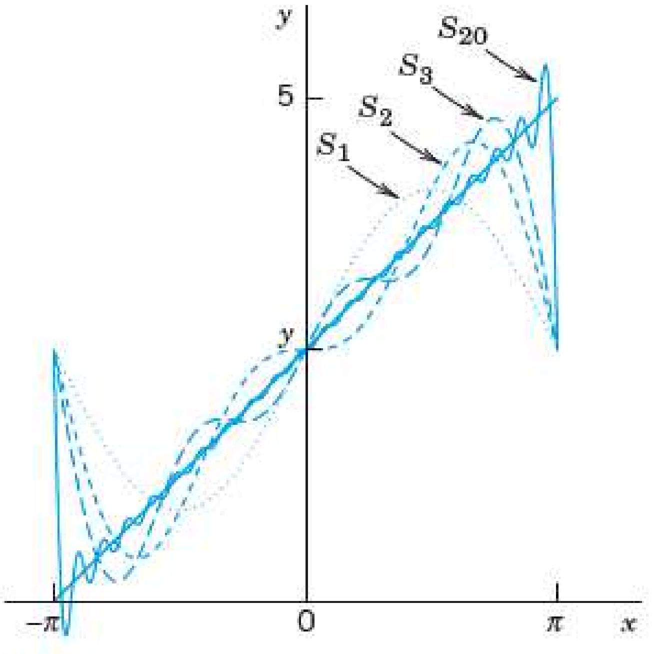


Fig. 269. Partial sums S_1, S_2, S_3, S_{20} in Example 5

Half Range Extension

Definition Let $f : [0, L] \to \mathbb{R}$.

1. Define $f_E : (-\infty, \infty) \to \mathbb{R}$ by

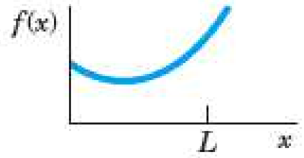
$$f_E(x) = \begin{cases} f(-x) & -L < x \leq 0 \\ f(x) & 0 < x \leq L \end{cases}, \quad f_E(x+2L) = f_E(x).$$

2. Define $f_O : (-\infty, \infty) \to \mathbb{R}$ by

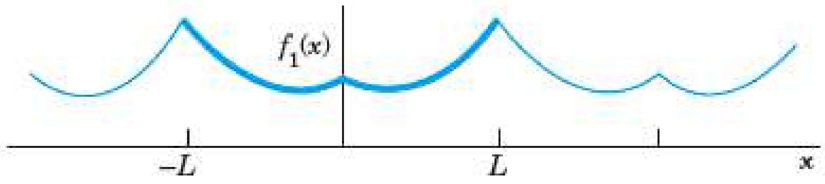
$$f_O(x) = \begin{cases} -f(-x) & -L < x \leq 0 \\ f(x) & 0 < x \leq L \end{cases}, \quad f_O(x+2L) = f_O(x).$$

Note. f_E is even $2L$-periodic and f_O is odd $2L$-periodic.

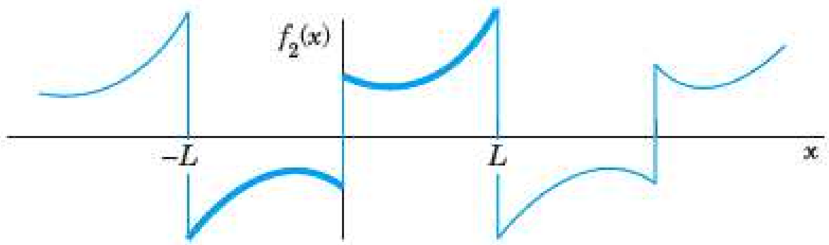
Half Range Extension



(0) The given function $f(x)$



(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Fourier Cosine Series

Since f_E is even $2L$-periodic, it has Fourier series

$$f_E(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (-L \leq x \leq L),$$

$$\text{where } a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad (n = 0, 1, 2, \ldots).$$

But f_E extends f, so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (0 \leq x \leq L).$$

This is called the **Fourier cosine series** of f.

Fourier Sine Series

Similarly, since f_O is odd $2L$-periodic, it has Fourier series

$$f_O(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (-L \leq x \leq L),$$

where $b_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L} dx.$

But f_O extends f, so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (0 \leq x \leq L).$$

This is called the **Fourier sine series** of $f : [0, L] \to \mathbb{R}$.

Example 7

EX. Let $f : [0, \pi] \to \mathbb{R}$ be given by

$$f(x) = x \quad (0 \leq x \leq \pi).$$

Find the Fourier sine series of $f(x)$.