Mock Exam

1. Answer the following questions.

1.1. $\Gamma\left(\dfrac{3}{2}\right) = \ldots\ldots\ldots \Gamma(n) = \ldots\ldots$ (n is a positive integer).

1.2. The general solution of $x^2y'' + xy' + (x^2 - 5)\,y = 0$ is $y = \ldots\ldots\ldots\ldots$

1.3. The general solution of $(1 - x^2)y'' - 2xy' + 6y = 0$ is $y = \ldots\ldots\ldots\ldots$

2. Use the *power series method* to find the series solution $y = \sum_{n=0}^{\infty} b_n x^n$ of the equation

$$y'' + xy' + y = 0.$$

Calculate the first 6 non-zero coefficients b_n's.

3. Use the *Frobenius method* to find a nonzero series solution $y = x^r \sum_{n=0}^{\infty} b_n x^n$ of the equation

$$4x^2y'' + 4xy' - y = 0 \quad (x > 0).$$

4. Evaluate the following Laplace transforms

4.1. $\mathcal{L}[2\sin 3t] \ldots\ldots\ldots\ldots, \mathcal{L}[\sqrt{t}] = \ldots\ldots\ldots\ldots\ldots$

4.2. $\mathcal{L}\left[e^{-t}t\right]$

4.3. Let $f(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ 6 & \text{if } 1 < t < 4 \\ 1 & t > 4 \end{cases}.$

(a) Express $f(t)$ in terms of the Heaviside function.

(b) Evaluate $\mathcal{L}\left[f(t)\right]$.

5. Evaluate the following inverse Laplace transforms

5.1. $\mathcal{L}^{-1}\left[\dfrac{3}{s+2}\right] = \ldots\ldots\ldots, \mathcal{L}^{-1}\left[\dfrac{5s}{s^2+1}\right] = \ldots\ldots\ldots$

5.2. $\mathcal{L}^{-1}\left[\dfrac{e^{-5s}}{s^2+9}\right]$

5.3. $\mathcal{L}^{-1}\left[\dfrac{s-3}{s^2-6s+10}\right]$

6. Use the Laplace transformation method to solve the initial value problem:

$$\begin{cases} y'' - y = \delta(t-3), \\ y(0) = 0,\ y'(0) = 0. \end{cases}$$

7. Let

$$f(x) = \sin^2(x).$$

7.1. Show that $f(x)$ is 2π-periodic.

7.2. Find the *Fourier series expansion* of $f(x)$.

8. Let $h : [0, 4] \to \mathbb{R}$ be a function defined by $h(x) = \begin{cases} 0 & \text{if } 0 \le x < 2, \\ -3 & \text{if } 2 < x \le 4. \end{cases}$

8.1. What is the odd extension of $h(x)$?

8.2. Expand the *Fourier sine series* of $h(x)$.

9. Find the *Fourier integral expansion* of the function $u(x) = \begin{cases} 2 & |x| < 3 \\ 0 & |x| > 3. \end{cases}$

10. Find the *Fourier transform* of the function $v(x) = \begin{cases} e^{-4x} & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$

11. Solve the IBVP

$$\begin{cases} u_{tt} = 4u_{xx} & 0 < x < \pi, t > 0, \\ u(0, t) = u(\pi, t) = 0 & t > 0, \\ u(x, 0) = f(x), \quad \dfrac{\partial u}{\partial t}(x, 0) = 0 & 0 < x < \pi. \end{cases}$$

12. Solve the BVP

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < 1, 0 < y < \pi, \\ u_x(x, 0) = u_x(x, \pi) = 0 & 0 < x < 1, \\ u(0, y) = 0, \ u(1, y) = y & 0 < y < 1. \end{cases}$$