3.4. Integration by Substitution

Consider the integral

$$\int (x^2 + 1)^6 \, dx$$

We want to employ the ***u*-substitution** technique:

$$u = x^2 + 1$$

as in the the calculation of derivative

$$\frac{d}{dx}[(x^2 + 1)^6]$$

For the latter, we have **the chain rule**

$$\frac{d}{dx}[f(u)] = \frac{df(u)}{du} u'$$

For $f(u) = u^6$, there is the **derivative formula**

$$\frac{du^n}{du} = nu^{n-1}$$

We also have the **integration formula**

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$$

Thus, it would be nice if somehow one can emply the u-substitution.

It is however not true that

$$\int (x^2+1)^6\,dx = \int u^6\,du$$

because LHS is integration in x while RHS is in u.

Like the chain rule for differentiation, there must be a term multiplied to $(x^2+1)^6$, so that we can transform the integral. It is $u' = du/dx$.

Integration by Substitution

For the integral

$$\int f\big(g(x)\big)g'(x)\,dx$$

If we put $u = g(x)$, so $u' = g'(x)$ then

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

RHS is understood with $u = g(x)$ subsituted back!

$$\text{RHS} = \int f(u)\,du\Big]_{u=g(x)}$$

So if $F(x)$ is an antiderivative of $f(x)$ then

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

Proof Let

$$H(u) + C = \int f(u)\, du$$

It means $H(u)$ is an antiderivative, i.e.

$$H'(u) = \frac{d}{du}[H(u)] = f(u)$$

Now

$$\left.\int f(u)\, du\right]_{u=g(x)} = H(g(x)) + C$$

By the chain rule

$$\frac{d}{dx}[H(g(x))] = H'(g(x))g'(x)$$

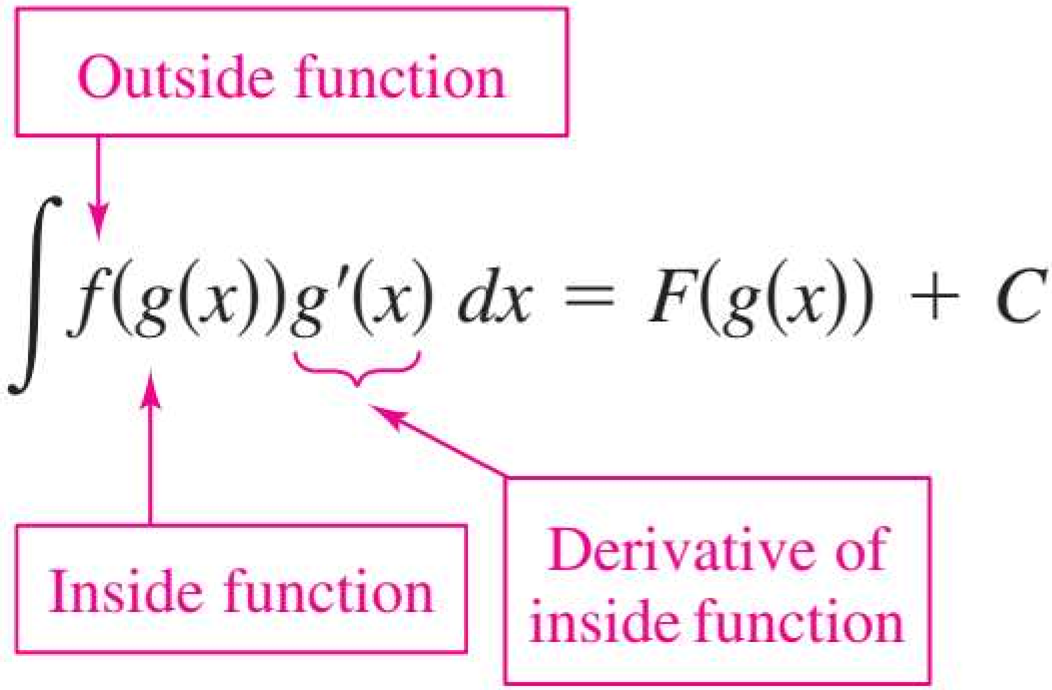
$$= f(g(x))g'(x)$$

I.e. $H(g(x))$ is an antiderivative of $f(g(x))g'(x)$:

$$H(g(x)) + C = \int f(g(x))g'(x)\, dx$$

Therefore,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du\Big]_{u=g(x)}$$



EXAMPLE. Evaluate the integral

$$\int (x^2 + 1)^6 (2x)\, dx$$

EXAMPLE. Evaluate the indefinite integral

$$\int 3\cos(3x)\,dx$$

EXAMPLE. Evaluate the integral

$$\int (\sec^2 x)(\tan x + 3)\, dx$$

EXAMPLE (Multiply/divide by constant). Find the integral

$$\int x^2(x^3+1)^{10}\,dx$$

EXAMPLE. Verify the integration formula

$$\int u^n u' \, dx = \frac{u^{n+1}}{n+1} + C$$

for $n \neq -1$, then calculate the following integrals

1. $\int \sqrt{2x-1} \, dx$

2. $\int \frac{1}{(3x+2)^4} \, dx$

3. $\int x^2(1+2x^3)^8\,dx$

4. $\int \frac{x}{\sqrt{x^2+1}}\,dx$

EXAMPLE (Change into u and du). Evaluate the integral

$$\int x\sqrt{2x-1}\,dx$$

EXAMPLE (Change into u and du). Evaluate the integral

$$\int \sqrt{\sin x} \cos x \, dx$$

EXAMPLE. Verify the integration formulas

$$\int (\cos u)u'\,dx = \sin u + C$$

$$\int (\sin u)u'\,dx = -\cos u + C$$

then employ the formula to calculate

1. $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

2. $\displaystyle\int x\sin(x^2)\,dx$

EXAMPLE. Find

1. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

2. $\int \sqrt[3]{\tan x}\,\sec^2 x\,dx$

To integrate

$$\int f(g(x))A(x)\,dx$$

where $A(x)$ may not exactly be $g'(x)$, we may use the **differential notation:** $\boldsymbol{u = g(x)}$

$$du = g'(x)dx \quad \Rightarrow \quad dx = \frac{1}{g'(x)}du$$

Then

$$\int f(g(x))A(x)\,dx = \int f(u)\frac{A(x)}{g'(x)}du$$

Solving $\frac{A(x)}{g'(x)} = B(u)$, then

$$\int f(g(x))\frac{A(x)}{g'(x)}dx = \int f(u)B(u)\,du$$

EXAMPLE. Evaluate the integral

$$\int \frac{2x+1}{\sqrt{x+4}}\,dx$$

EXAMPLE. Find

$$\int \frac{x^2 + x^5}{\sqrt{1 - x^6}} dx$$

EXAMPLE. Evaluate the integral

$$\int \frac{1}{x^5 + x^{-3} + 2x}\,dx$$