Method of Applied Math
Lecture 1: Series Solutions of Second Order Linear ODEs

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We solve second order linear ODE

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where $p(x), q(x)$ are functions of a variable x by the **power series method.**

Recall

$$y' = \frac{dy}{dx} = \text{the first derivative of } y,$$
$$y'' = \frac{d^2y}{dx^2} = \text{the second derivative of } y.$$

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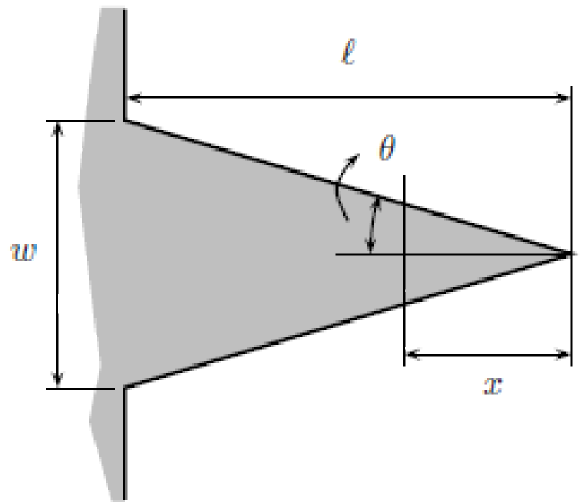
Example 1 (Wedge heat sink)

EX. The model equation for a wedge heat sink



is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\mu xy=0$$

where $y(x)$ is the temperature at x and $\mu > 0$ is a constant.

Example 2 (Beam under loads)

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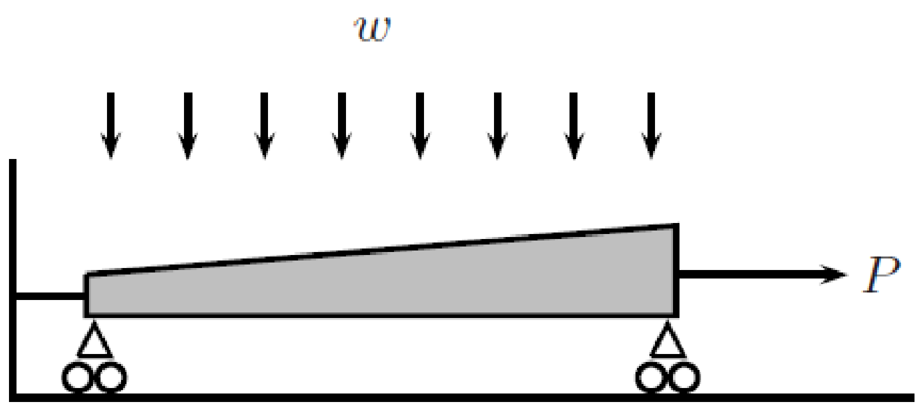
EX. The model equation for a beam of length $2L$ under loads



is

$$ax\frac{d^2y}{dx^2} - Py = \frac{1}{2}wx^2 - wx$$

where $y(x)$ is the deflection at x and $a > 0$ is a constant.

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Plan.

1. Power series
2. Ordinary points v.s. Regular singular points
3. Power series solutions near an ordinary point.

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A **power series** is an expression of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots ,$$

c_k are called **coefficients** and a is called the **center**.

The sum, where the series converges, defines a function of x, i.e.

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k.$$

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EX. The power series

$$3 - 6x + 12x^2 - 24x^3 + \cdots$$

has center: $a = 0$ and coefficients:

$$c_0 = 3, c_1 = -6, c_2 = 12, \dots, c_k = 3 \cdot (-2)^k, \dots$$

It sum is the function

$$\begin{aligned} f(x) &= 3 - 6x + 12x^2 - 24x^3 + \cdots \\ &= 3(1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots) \\ &= 3\sum_{k=0}^{\infty}(-2x)^k = \frac{3}{1+2x} \end{aligned}$$

Taylor Series

An important group of power series is the Taylor series of differentiable functions.

Definition. Let f be a differentiable function. The **Taylor series** of $f(x)$ about a is

$$T_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

or

$$T_a(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

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EX. Let $f(x) = e^x$. Since

$$(e^x)' = e^x,$$

we have

$$f'(0) = 1, \quad f''(0) = 1, \quad \ldots f^{(k)}(0) = 1 \quad \forall\, k.$$

So the Taylor series of $f(x) = e^x$ about 0 is

$$T_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Analytic Functions

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For a differentiable function f, if its Taylor series at a converges on an interval $I = (a - R, a + R)$, then

$$f(x) = T_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \forall\, x \in I.$$

We say that f is **analytic** at a.

Fact. Most functions are analytic on their **domains**.

- e^x is analytic at any $a \in (-\infty, \infty)$.
- A polynomial $P(x)$ is analytic at any $a \in (-\infty, \infty)$.
- A rational function $\dfrac{P(x)}{Q(x)}$ is analytic at any a where $Q(a) \neq 0$.

Ordinary and Singular Points

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Definition. Consider the ODE

$$y'' + p(x)y' + q(x)y = f(x).$$

If p, q, f are analytic at a, then a is called an **ordinary point** of the ODE.

If p, q, or f is not analytic at a, then a is called a **singular point** of the ODE.

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EX. For the ODE

$$3y'' + 4xy' - 5x^2y = 0$$

find all ordinary points and singular points of this equation.

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EX. For the ODE

$$(x^2 - 1)y'' + xy' - y = 0$$

find all ordinary points and singular points of this equation.

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Series Solutions: An Ordinary Point

At an ordinary point, we have the following theorem.

Theorem. [Kreyszig, 5.1 Theorem 1] Let a be an **ordinary point** of the ODE

$$y'' + p(x)y' + q(x)y = f(x).$$

Then any solution of the ODE is analytic at a, so

$$y = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots = \sum_{k=0}^{\infty} b_k(x-a)^k,$$

for some constants $b_0, b_1, b_2, \ldots$

We will pay a special attention to the case $a = 0$ and $f(x) = 0$.

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Series Solutions: An Ordinary Point

Thus, solutions to $y'' + p(x)y' + q(x)y = 0$ have the form of power series, about each ordinary point a.

Method. **Step 1.** Set the solution y as a series about a

$$y = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots ,$$

that is

$$y = \sum_{k=0}^{\infty} b_k(x-a)^k$$

Step 2. Termwise differentiation:

Series Solutions: An Ordinary Point

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$$y' = b_1 + b_2(2)(x-a) + b_3(3)(x-a)^2 + \cdots$$
$$y'' = b_2(2)(1) + b_3(3)(2)(x-a) + b_4(4)(3)(x-a)^2 + \cdots$$

that is

$$y' = \sum_{k=0}^{\infty} b_{k+1}(k+1)(x-a)^k$$
$$y'' = \sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)(x-a)^k.$$

Step 3. Solve for $b_0, b_1, b_2, \ldots$

Example 7 (Airy's equation)

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EX. Show that 0 is an ordinary point of the ODE

$$y'' - xy = 0.$$

Then find the general solution of the ODE as power series about 0.

Sol. It is easy to verify that $a = 0$ is an ordinary point.

According to the series solution method, we set

$$y = \sum_{k=0}^{\infty} b_k x^k$$

$$y'' = \sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)x^k$$

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$$\sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)x^k - x\sum_{k=0}^{\infty} b_k x^k = 0$$

$$\sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} b_k x^{k+1} = 0$$

Shift the index the second series **to align the powers of** x $(j = k+1)$:

$$\sum_{k=0}^{\infty} b_k x^{k+1} = \sum_{j=1}^{\infty} b_{j-1}x^j \overset{dummy}{=} \sum_{k=1}^{\infty} b_{k-1}x^k$$

(we throw away j finally, and re-use k).

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Thus

$$\sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} b_{k-1}x^k = 0$$

Split terms in the first series (**to align** the starting sum):

$$b_2 \cdot 2 \cdot 1 + \sum_{k=1}^{\infty} b_{k+2}(k+2)(k+1)x^k$$

So

$$b_2 \cdot 2 \cdot 1 + \sum_{k=1}^{\infty} b_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} b_{k-1}x^k = 0$$

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Combine series

$$b_2 \cdot 2 \cdot 1 + \sum_{k=1}^{\infty} \left[b_{k+2}(k+2)(k+1) - b_{k-1} \right] x^k = 0$$

Note. $\sum_{k=0}^{\infty} c_k (x-a)^k = 0$ then $c_k = 0$ for all k. So

$$2b_2 = 0, \quad b_{k+2}(k+2)(k+1) - b_{k-1} = 0 \quad \text{for all } k \geq 1.$$

Thus

$$b_2 = 0, \quad b_{k+2} = \frac{b_{k-1}}{(k+1)(k+2)} \quad \text{for all } k \geq 1.$$

These equations are **recurrence equations**.

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Solve recurrence equations

- $b_2 = 0, \quad b_5 = 0, \quad b_8 = 0, \quad b_{11} = 0, \ldots$

  $b_{3n+2} = 0 \quad (n \geq 4)$

- $b_3 = \dfrac{b_0}{2 \cdot 3}, \quad b_6 = \dfrac{b_0}{2 \cdot 3 \cdot 5 \cdot 6}, \quad b_9 = \dfrac{b_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$

  $$b_{3n} = \frac{b_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)} \quad (n \geq 4)$$

- $b_4 = \dfrac{b_1}{3 \cdot 4}, \quad b_7 = \dfrac{b_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad b_{10} = \dfrac{b_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$

  $$b_{3n+1} = \frac{b_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)} \quad (n \geq 4)$$

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Set back to the solution:

$$y = \sum_{k=0}^{\infty} b_k x^k = b_0 \left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots\right] + b_1 \left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots\right]$$

This is the general solution where the fundamental solutions are

$$y_1 = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots ,$$

$$y_2 = x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots$$