Method of Applied Math
Lecture 3: Frobenius Method

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Series Solutions: Regular Singular Points

Now we solve the equation

$$y'' + p(x)y' + q(x)y = 0$$

when a is a singular point. This means $p(x)$ or $q(x)$ fails to be analytic at $x = a$!

- a is a **regular singular point**.
- **Frobenius method**.
- Application to the **Bessel equation**.

EX. Show that $a = 0$ is a singular point for each of the following equations:

$$x(1-x)y'' + (4-3x)y' - y = 0$$
$$xy'' + (5-x)y' + 2y = 0$$
$$x^2y'' + x(x - \frac{1}{2})y' + \frac{1}{2}y = 0$$

Sol.

Motivation

EX. Consider a Cauchy-Euler equation $x^2y'' + 3xy' + \frac{3}{4}y = 0$.

- Verify that $y_1 = 1/\sqrt{x}$, $y_2 = x^{-3/2}$ are solutions.
- Show that if we apply the power series method about $a = 0$, we get the trivial solution $y = 0$!

Sol.

Regular Singular Point

Defition. Let a be a singular point for the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

a is called a **regular singular point** for this ODE if

$$(x-a)p(x) \text{ and } (x-a)^2 q(x) \text{ are analytic at } x = a.$$

On the other hand, if

$$(x-a)p(x) \text{ or } (x-a)^2 q(x) \text{ fails to be analytic at } x = a,$$

we call a an **irregular singular point** for the ODE.

Example 1

EX. Consider the ODE $x^2(x-1)y'' + y' - y = 0$.

- Show that 1 is a regular singular point for the equation.
- Show that 0 is an irregular singular point for the equation.

Sol.

Example 2

EX. Consider the Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Show that 0 is a regular singular point of the ODE.
Sol.

Example 3

EX. Show that $a = 0$ is a regular singular point for each of the following equations:

$$x(1-x)y'' + (4-3x)y' - y = 0$$
$$xy'' + (5-x)y' + 2y = 0$$
$$x^2y'' + x(x - \frac{1}{2})y' + \frac{1}{2}y = 0$$
$$x^2y'' + 3xy' + \frac{3}{4}y = 0.$$

Sol.

Frobenius Method

Theorem (Frobenius). Let a be a **regular singular point** for

$$y'' + p(x)y' + q(x)y = 0.$$

Then there is a constant r such that the ODE has a solution of the form

$$\begin{aligned} y &= (x-a)^r \sum_{n=0}^{\infty} b_n (x-a)^n \\ &= b_0(x-a)^r + b_1(x-a)^{r+1} + b_2(x-a)^{r+2} + \cdots . \end{aligned}$$

The process of obtaining a solution of the above form is called the **Frobenius method**.

Example 4

EX. We have already shown that $a = 0$ is a regular singular point for the equation $x^2y'' + 3xy' + \frac{3}{4}y = 0$. Find all possible constants r according the Frobenius theorem.

Sol. Plugging Frobenius solution into ODE. The terms having the smallest power are $b_0r(r-1)x^r + 3b_0rx^r + \frac{3}{4}b_0x^r$.So

$$b_0(r(r-1) + 3r + \frac{3}{4}) = 0.$$

Thus

$$r_1 = -\frac{1}{2}, \quad r_2 = -\frac{3}{2}.$$

Frobenius Method

Theorem. Let a be a regular singular point for the ODE

$$y'' + p(x)y' + q(x)y = 0.$$

If r is the **larger root** of the quadratic equation

$$r(r-1) + p_0 r + q_0 = 0 \qquad \textbf{(indicial equation)}$$

where

$$p_0 = \lim_{x\to a}(x-a)p(x), \quad q_0 = \lim_{x\to a}(x-a)^2 q(x),$$

then the Frobenius theorem is true.

Example 5

EX. Consider the ODE $x^2y'' - 3xy' + 3y = 0$.

- Find a series solution using the Frobenius method about $a = 0$.
- Use reduction of order to find the other linearly independent solution.

Sol.

Example 6

EX. For the ODE $x^2y'' + xy' + (x-1)y = 0$, using the Frobenius method about $a = 0$ we get a series solution $y = x^r \sum_{k=0}^{\infty} b_k x^k$. Calculate the series up to b_3.

Sol. Larger root $r = 1$,

$$y_1 = b_0 \left(x - \frac{1}{3}x^2 + \frac{1}{24}x^3 + \cdots \right).$$

Example 7

EX. For the ODE $xy'' - 3y' - y = 0$, use the Frobenius method about $a = 0$ to find a series solution with coefficients up to b_3.

Sol. Larger root $r = 4$.

$$y = b_0 \left(\frac{1}{5}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 + \cdots \right)$$