Method of Applied Math

Lecture 4: Bessel Equation

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Bessel Equation

Let ν be a non-negative constant. The equation of the form

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

is called the **Bessel equation** of order ν.

Note that $a = 0$ is a regular singular point. Expressing in the standard form

$$y'' + \frac{1}{x}y' + \frac{x^2 - \nu^2}{x^2}y = 0 \quad \Rightarrow \quad p(x) = \frac{1}{x}, q(x) = \frac{x^2 - \nu^2}{x^2}$$

we have

$$p_0 = \lim_{x \to 0} xp(x) = 1, \qquad q_0 = \lim_{x \to 0} x^2q(x) = -\nu^2.$$

Bessel Equation

The indicial equation is

$$r(r-1)+r-\nu^2=0$$

or

$$r^2-\nu^2=0 \quad \Rightarrow \quad r_1=\nu, \quad r_2=-\nu.$$

The larger root is $r_1=\nu$.

So by Frobenius' theorem, the Bessel equation has a series solution

$$\begin{aligned} y &= x^{\nu}(b_0+b_1x+b_2x^2+\cdots) \\ &= b_0x^{\nu}+b_1x^{\nu+1}+b_2x^{\nu+2}+\cdots . \end{aligned}$$

Example 1

EX. Find a Frobenius series solution for the Bessel equation of order 1:

$$x^2y'' + xy' + (x^2 - 1)y = 0,$$

by calculating the first four coefficients.

Sol. $y = b_0x + b_1x^2 + b_2x^3 + b_3x^4 + b_4x^5 + \cdots$

$$xy' = b_0x + 2b_1x^2 + 3b_2x^3 + 4b_3x^4 + 5b_4x^5 + \cdots$$

$$x^2y'' = 2b_1x^2 + 6b_2x^3 + 12b_3x^4 + 20b_4x^5 + \cdots$$

$$x^1: \quad b_0 - b_0 = 0$$
$$x^2: \quad 2b_1 + 2b_1 - b_1 = 0 \quad \Rightarrow b_1 = 0$$

Example 1

$$x^3: \quad 6b_2 + 3b_2 + b_0 - b_2 = 0 \quad \Rightarrow b_2 = -\frac{1}{8}b_0$$
$$x^4: \quad 12b_3 + 4b_3 + b_1 - b_3 = 0 \quad \Rightarrow b_3 = 0$$
$$x^5: \quad 20b_4 + 5b_4 + b_2 - b_4 = 0 \quad \Rightarrow b_4 = -\frac{1}{24}b_2 = \frac{1}{192}b_0$$

So we get

$$y = b_0\left(x - \frac{1}{8}x^3 + \frac{1}{192}x^5 + \cdots\right).$$

Taking $b_0 = \frac{1}{2}$, we can write the solution as

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu} \quad (\nu = 1).$$

Solving Bessel equation

Generally, we have the following solution for the Bessel equation of order ν.

Theorem. The function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu},$$

is a solution of the Bessel equation of order ν.

This solution is obtained by taking the larger root of the indicial equation.

Solving Bessel equation

Frobenius' method. Choose $a = 0$ and use the larger root $r = \nu$ of the indicial equation.

Step 1 Set

$$y = \sum_{k=0}^{\infty} b_k x^{k+\nu}$$

$$y' = \sum_{k=0}^{\infty} b_k (k+\nu) x^{k+\nu-1}$$

$$y'' = \sum_{k=0}^{\infty} b_k (k+\nu)(k+\nu-1) x^{k+\nu-2}.$$

Solving Bessel equation

Step 2. Recurrence equations:

$$k = 0 : \nu(\nu - 1)b_0 + \nu b_0 - \nu^2 b_0 = 0$$
$$k = 1 : (\nu + 1)\nu b_1 + (\nu + 1)b_1 - \nu^2 b_1 = 0$$
$$k \geq 2 : (k + \nu)(k + \nu - 1)b_k + (k + \nu)b_k + b_{k-2} - \nu^2 b_k = 0$$

Step 3 Solve the recurrence equations to get

$$b_1 = b_3 = b_5 = \cdots = 0$$

and

$$b_{2n} = \frac{(-1)^n b_0}{2^{2n} n!(\nu + 1)(\nu + 2)\cdots(\nu + n)}, \qquad n = 1, 2, \ldots$$

Step 4 Let

$$b_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \quad \Rightarrow \quad b_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(\nu+n+1)}$$

The solution is

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

Bessel Functions of the First Kind

Definition. The function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

is called the **Bessel function of the first kind** of order ν.

Recall. Γ is the Gamma function. It has the properties that

$$\Gamma(n + 1) = n! \quad n \in \{0, 1, 2, \ldots\},$$

and

$$\Gamma(x + n + 1) = (x + n)(x + n - 1) \cdots x\Gamma(x).$$

Example 2

EX. Use the solution formula to find a solution of the equation

$$xy'' + y' + xy = 0 \quad (x > 0).$$

Sol.

Example 3

EX. Show that the following property hold:

$$(x^\nu J_\nu(x))' = x^\nu J_{\nu-1}(x).$$

Sol.

Bessel Functions of the Second Kind

The indicial equation of the Bessel equation of order ν has two roots $r_1 = \nu, r_2 = -\nu$.

The larger root $r_1 = \nu$ leads to a solution $y_1 = J_\nu(x)$.

For the root $r_2 = -\nu$, one get another solution

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-\nu + k + 1)} \left(\frac{x}{2}\right)^{2k-\nu} .$$

A difficulty: $\{J_\nu, J_{-\nu}\}$ are linearly dependent for $\nu = 0, 1, 2, \ldots$ In fact, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

If $\nu \notin \{0, 1, 2, \ldots\}$ then the Bessel equation has a general solution

$$y = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

Example 4

EX. Show that $J_{-1}(x) = -J_1(x)$.

Sol.

Example 5

EX. Find the general solution of the Bessel equation

$$x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0.$$

Sol.

Bessel Functions of the Second Kind

Definition. We define

$$Y_\nu(x) = \frac{1}{\sin \nu\pi} \left[J_\nu(x) \cos \nu\pi - J_{-\nu}(x) \right], \quad \nu \notin \{0, 1, 2, \ldots\}$$

and

$$Y_\nu(x) = \lim_{\mu \to \nu} Y_\mu(x), \quad \nu \in \{0, 1, 2, \ldots\}.$$

Y_ν is another solution to the Bessel equation of order ν and

$$\{J_\nu, Y_\nu\} \quad \text{are fundamental solutions.}$$

$Y_\nu(x)$ is called **Bessel function of the second kind** order ν.

Theorem. The complete solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

is

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

where C_1, C_2 are constants.

Example 6

EX. Find the complete solution of the Bessel equation

$$x^2y'' + xy' + \left(x^2 - 5\right) y = 0.$$

Sol.

Example 7

EX. Find the complete solution of the equation

$$y'' + \left(e^{-2x} - \frac{1}{9}\right) y = 0$$

by changing the variable $z = e^{-x}$.

EX.