Mathematical Analysis: Lecture 1

Sujin Khomrutai, Ph.D.

Notation

1. $\mathbb{N}$ = the set of positive integers, $\mathbb{Z}$ = the set of integers.
2. $\mathbb{Q}$ = the set of rational numbers, i.e. all m/n where $m, n \in \mathbb{Z}, n \neq 0$.
3. $\mathbb{R}$ = the set of real numbers.
4. $x \in A$ for "x is a member of A", $A \subset B$ for A is a subset of B.

We assume the reader is familiar with the number systems $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$.

Summary. *We study the real number system* $\mathbb{R}$ *including*

1. *algebraic property of* $\mathbb{R}$*,*
2. *ordered structure of* $\mathbb{R}$*.*
3. *bounded subsets of ordered field.*

1 The Real Number System

The number system $\mathbb{Q}$ is *algebraically closed* in the sense that if $x, y \in \mathbb{Q}$ then $x + y, xy \in \mathbb{Q}$. In particular, $x^2 = x \cdot x \in \mathbb{Q}$. However, it is not *analytically closed* in a very simple circumstance that

$$x^2 = 2 \quad \text{has no solution } x \in \mathbb{Q}.$$

Since solving equation is a basic and essential tool in real-life problem, it is necessarily to have a number system so that one can do such basic tasks. This leads to the real number system $\mathbb{R}$ that will be discussed throughout this course.

In the first two lectures, we study the real number system. We will not use the "bottom-up" approach, i.e.

1. constructing $\mathbb{N}$ using Peano Axioms,
2. introducing negative numbers to get $\mathbb{Z}$ (so every equation $x + m = n$ can be solved),
3. including fractions $m/n(n \neq 0)$ to get $\mathbb{Q}$ (so every equation $ax + b = 0$ can be solved).
4. introducing irrational numbers to get $\mathbb{R}$.

We shall use the "top-down" approach, that is describing $\mathbb{R}$ by properties and axioms.

There are other approaches to define real number such as $\sqrt{2}$ (the positive solution of $x^2 = 2$) as the limit to the sequence of rational numbers

$$\{1, \quad 1.4, \quad 1.41, \quad 1.414, \quad \ldots\}.$$

This is the construction of real numbers by *Cauchy sequences.*

We shall recall key properties of $\mathbb{R}$ and then apply to see further properties obtained from them e.g. concepts of continuity, differentiation, integration and etc.

1.1 Algebraic properties of $\mathbb{R}$

Axiom ($\mathbb{R}$ as a field). On $\mathbb{R}$, there are two binary operations called *addition* $(+)$ and *multiplication* $(\cdot)$ such that the following axioms hold.

(A1) $x + y = y + x$

(A2) $(x + y) + z = x + (y + z)$

(A3) $\exists\, 0 \in \mathbb{R}$ such that $0 + x = x = x + 0$.

(A4) $\forall\, x \in \mathbb{R}, \exists - x \in \mathbb{R}$ such that $x + (-x) = 0 = (-x) + x$.

(M1) $x \cdot y = y \cdot x$

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M3) $\exists\, 1 \in \mathbb{R}$ such that $1 \cdot x = x = x \cdot 1$.

(M4) $\forall\, x \in \mathbb{R}, x \neq 0, \exists\, 1/x \in \mathbb{R}$ such that $x \cdot (1/x) = 1 = (1/x) \cdot x$.

(D) $(x + y) \cdot z = x \cdot z + y \cdot z$.

Remark.

(a) From (A1)-(A4), we get that $\mathbb{R}$ is an Abelian group under the addition, and the same for $\mathbb{R} - \{0\}$ under the multiplication via (M1)-(M4). (D) is called the distributive law.

(b) One usually write

$$x - y, \quad \frac{x}{y}, \quad x + y + z, \quad xyz, \quad x^2, \quad x^3, \quad 2x, \quad 3x, \ldots$$

instead of

$$x{+}(-y), \quad x{\cdot}(1/y), \quad (x{+}y){+}z, \quad (x{\cdot}y){\cdot}z, \quad x{\cdot}x, \quad x{\cdot}x{\cdot}x, \quad x{+}x, \quad x{+}x{+}x, \ldots$$

(c) $\mathbb{Q}$ enjoy the same properties and is a field. In fact, $\mathbb{Q}$ is a subfield of $\mathbb{R}$.

Theorem 1.1. *The following statements are true.*

1. *If $x + y = x + z$ then $y = z$.*
2. *If $x + y = x$ then $y = 0$.*
3. *If $x + y = 0$ then $y = -x$.*
4. *$-(-x) = x$.*
5. *If $x \neq 0$ and $x \cdot y = x \cdot z$ then $y = z$.*
6. *If $x \neq 0$ and $x \cdot y = x$ then $y = 1$.*
7. *If $x \neq 0$ and $x \cdot y = 1$ then $y = 1/x$.*
8. *If $x \neq 0$ then $1/(1/x) = x$.*

Proof. 1. We have

$$\begin{aligned}
&(-x) + (x + y) = (-x) + (x + z) \\
&((-x) + x) + y = ((-x) + x) + z \\
&0 + y = 0 + z \\
&\therefore \; y = z.
\end{aligned}$$

2. This follows from (1) by taking $z = 0$.

3. This is true from (1) by using $z = (-x)$.

4. Since $(-x) + x = 0$, we get from (3) that $x = -(-x)$.

5.–8. are left as an exercise. □

Question. How to prove a statement "$p \Rightarrow q$"?

Question. Similarly, how to prove "$p \Leftrightarrow q$"?

Exercise 1.1.

(1) Let $x, y \in \mathbb{R}$. Prove the following statements.

(a) $0x = 0$ *(b) If* $x \neq 0$ *and* $y \neq 0$ *then* $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$. *(d)* $(-x)(-y) = xy$

(2) Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Prove that

$$x^n - y^n = (x-y)\sum_{k=1}^{n-1} x^{n-k}y^{k-1}.$$

(3) Show that if $x, y \in \mathbb{Q}$ then $x + y, xy \in \mathbb{Q}$.

(4) Show that if $x \in \mathbb{Q}$ and y is an irrational number, then $x + y$ is an irrational number. If in addition, $x \neq 0$ then xy is an irrational number.

1.2 Order structure of $\mathbb{R}$

Axiom (Ordered field axioms). There is a subset of real numbers $\mathbb{R}_+$, where we write $x > 0$ for each element $x \in \mathbb{R}_+$, such that the following properties hold:

(O1) If $x > 0$ and $y > 0$ then $x + y > 0$ and $xy > 0$.

(O2) For a real number x, exactly one of the following three alternatives is true:

$$x > 0, \quad -x > 0, \quad x = 0.$$

Definition 1.1. Let $x, y \in \mathbb{R}$. If $y - x > 0$, we write

$$x < y \quad \text{or} \quad y > x,$$

and we say that x is *less than* y or y is *greater than* x.

If either $x = y$ or $y - x > 0$, we write

$$x \leqslant y \quad \text{or} \quad y \geqslant x$$

and we say that x is *less than or equal to* y or y is *greater than or equal to* x.

Proposition 1. *The following properties are valid.*

1. $x < y \Leftrightarrow -x > -y$.
2. *If* $x < y$ *and* $y < z$ *then* $x < z$.

3. *If $x < y$ and $z < w$ then $x + z < y + w$.*

4. *If $x < y$ and $z > 0$ then $xz < yz$.*

5. *For $x, y \in \mathbb{R}$ the following trichotomy is true: $x = y$, $x < y$, or $x > y$.*

6. *If $x \neq 0$ then $x^2 > 0$.*

Proof. 1. That $x < y$ means $y - x > 0$. But $(-x) - (-y) = y - x$, so $(-x) - (-y) > 0$, which means $-x > -y$.

2. That $x < y$, $y < z$ mean $y - x > 0$, $z - y > 0$. So $z - x = (z - y) + (y - x) > 0$ by (O1).

3. Similar to 2.

4. That $x < y$ means $y - x > 0$. Since $y - x > 0$ and $z > 0$, we get by (O1) that $yz - xz = (y - x)z > 0$. So $xz < yz$.

5. Applying (O2) to the real number $y - x$, we get that exactly one of the following statements is true: $y - x > 0$, $-(y - x) > 0$, or $y - x = 0$.

6. Suppose $x \neq 0$. By (O2), either $x > 0$ or $-x > 0$. If $x > 0$, then we have by (O1) that $x^2 = x \cdot x > 0$. If $-x > 0$, we have $x^2 = (-x)(-x) > 0$. □

Question. Why $1 > 0$?

Question. Why the following statement is true

$$\text{“If } x \leqslant y \text{ and } y \leqslant x \text{ then } x = y\text{”?}$$

Next, we recall the absolute value function

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

One can verify the following identities directly by cases

- $|x| \geqslant 0$ with equality if and only if $x = 0$.
- $|-x| = |x|$
- $-|x| \leqslant x \leqslant |x|$.
- $|xy| = |x||y|$.
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ $(y \neq 0)$.

Question. Why the statement “If $x \leqslant y$ and $-x \leqslant y$, then $|x| \leqslant y$” is true?

Proposition 2. *The following triangle inequalities hold.*

1. *$|x + y| \leqslant |x| + |y|$.*

2. $||x| - |y|| \leqslant |x - y|$.

Proof. 1. We have $\pm x \leqslant |x|$ and $\pm y \leqslant |y|$ so $\pm(x + y) \leqslant |x| + |y|$. This implies $|x + y| \leqslant |x| + |y|$.

2. It suffices to show that $\pm(|x| - |y|) \leqslant |x - y|$. By the preceding statement, we have $|x| = |(x - y) + y| \leqslant |x - y| + |y|$, so $|x| - |y| \leqslant |x - y|$. Similarly, $|y| = |(y - x) + x| \leqslant |x - y| + |x|$, so $-(|x| - |y|) \leqslant |x - y|$. □

Exercise 1.2.

1. *If* $0 < x < y$ *and* $0 < a < b$, *prove that* $0 < ax < by$.
2. *If* $x < y < 0$ *and* $a < b < 0$, *prove that* $0 < by < ax$.
3. *If* $a < b + \varepsilon$ *for all* $\varepsilon > 0$, *prove that* $a \leqslant b$.
4. *If* $0 < a \leqslant bx$ *for all* $x > 1$, *prove that* $a \leqslant b$.
5. *Prove that* $|x - z| \leqslant |x - y| + |y - z|$.
6. *Prove that* $|x - L| < \varepsilon$ *if and only if* $L - \varepsilon < x < L + \varepsilon$.

■

Definition 1.2. Let $\mathbb{F}$ be an ordered field (a field satisfying (O1) and (O2)) and $E \subset \mathbb{F}$ a non-empty set.

1. If $u \in \mathbb{F}$ satisfies $x \leqslant u$ for all $x \in E$, u is called an **upper bound** of E and E is said to be **bounded above**.
2. If $l \in \mathbb{F}$ satisfies $l \leqslant x$ for all $x \in E$, l is called a **lower bound** of E and E is said to be **bounded below**.
3. E is said to be **bounded** if it is both bounded above and bounded below.
4. A set E which is not bounded is called **unbounded**.

Example 1.1. On the field $\mathbb{Q}$, consider the set $E = \{x \in \mathbb{Q} : x^2 < 2\}$.

For every $x \in E$, we have

$$x^2 < 2^2 \quad \Rightarrow \quad |x| < 2.$$

This implies

$$-2 < x, \quad x < 2.$$

So E is bounded above and bounded below. Therefore E is a bounded set. ■

Definition 1.3. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ a non-empty set.

(1) Suppose $u^* \in \mathbb{F}$ satisfies

  (i) u^* is an upper bound of E, and

  (ii) $u^* \leqslant u$ for any upper bound u of E.

  Then u^* is called the **least upper bound** or the **supremum** of E, denoted

$$u^* = \sup E.$$

(2) Suppose $l^* \in \mathbb{F}$ satisfies

  (i) l^* is a lower bound of E, and

  (ii) $l^* \geqslant l$ for any lower bound l of E.

  Then l^* is called the **greatest lower bound** or the **infimum** of E, denoted

$$l^* = \inf E.$$

Question. If the supremum of E exists, should it be unique? Same for the infimum.

Mathematical Analysis: Lecture 2

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we talk about*

1. *the supremum and infimum of a set,*
2. *the completeness property of $\mathbb{R}$,*
3. *some applications of the completeness property of $\mathbb{R}$.*

Definition 1.4. Let E be a non-empty subset of $\mathbb{F}$. If $\sup E$ exists and belongs to E, it is also called the *maximum* of E and is denoted by $\max E$.

If $\inf E$ exists and belongs to E, it is called the *minimum* of E and is denoted by $\min E$.

Example 1.2. *The set $E = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded by Example 1.1 has neither supremum nor infimum in $\mathbb{Q}$. This will be proved later.* ■

Example 1.3. Let $E = \{x \in \mathbb{R} : a < x \leqslant b\}$ where $a < b$ are real numbers. Find $\sup E$ and $\inf E$.

Solution. The set of upper bounds of E is $[b, \infty)$, so the least-upper-bound is $\sup E = b$. Similarly, the set of lower bounds of E is $(-\infty, a]$, so $\inf E = a$.

Observe that $\sup E = b \in E$ whereas $\inf E = a \notin E$. ■

Remark. The result of the preceding example can be generalized as follows. Let $a < b$ be real numbers and E denote any one of the following sets

$$[a, b], \quad (a, b], \quad [a, b), \quad (a, b).$$

Then

$$\sup E = b \quad \text{and} \quad \inf E = a.$$

Example 1.4. Let $E = \{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$. Find $\sup E$ and $\inf E$.

Solution. We solve the inequality $x^2 - 5x + 6 < 0$ to get

$$x^2 - 5x + 6 < 0 \quad \Leftrightarrow \quad (x-2)(x-3) < 0 \quad 2 < x < 3.$$

So $E = (2, 3)$ hence as above $\inf E = 2, \sup E = 3$. ∎

Question. Use induction to prove the identity

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

Question. If E is a non-empty, finite subset of an ordered field $\mathbb{F}$, what can we say about the supremum and infimum of E?

Proposition 3 (Approximation property)**.** *Let E be a non-empty subset of $\mathbb{F}$.*

1. *If $\sup E$ exists, then for each $r \in \mathbb{F}$ with $r < \sup E$, $\exists\ x \in E$ such that $r < x \leqslant \sup E$.*

2. *If $\inf E$ exists, then for each $r \in \mathbb{F}$ with $r > \inf E$, $\exists\ x \in E$ such that $\inf E \leqslant r < x$.*

Proof. 1. Since $r < \sup E$, r is not an upper bound of E. So there is $x \in E$ such that $r < x$. The proof of second statement is similar. □

Remark. The converse to Proposition 3 is also true.

1. If there is $u \in \mathbb{F}$ such that for any $r < u$, $\exists\ x \in E$ such that $r < x \leqslant u$, then $\sup E = u$.

2. If there is $l \in \mathbb{F}$ such that for any $r > l$, $\exists\ x \in E$ such that $l \leqslant x < r$, then $\inf E = l$.

Observe that if $A = [a, b]$ and $B = [c, d]$ are two intervals and suppose $A \subset B$, then $a \geqslant c$ and $b \leqslant d$. This implies $\inf A \geqslant \inf B$ and $\sup A \leqslant \sup B$.

Example 1.5. Let $A \subset B$ be non-empty bounded subset of $\mathbb{F}$. Prove that $\sup A \leqslant \sup B$ and $\inf A \geqslant \inf B$.

Solution. Let $r = \sup A, s = \sup B$. First, we show that s is an upper bound of A, that is $a \leqslant s$ for all $a \in A$. For any $a \in A$, since $A \subset B$, we have $a \in B$. Since s is an upper bound of B, it follows that $a \leqslant s$.

Now s is an upper bound of A, so we obtain $\sup A \leqslant s$, i.e. $\sup A \leqslant \sup B$. For the second inequality, see the question below. ∎

Question. Prove that $\inf A \geqslant \inf B$.

1.3 The completeness property of $\mathbb{R}$

We begin with the following question.

Question. We denote $-E = \{-x : x \in E\}$. Show that

$$\inf(-E) = -\sup E \quad \text{and} \quad \sup(-E) = -\inf E.$$

Now we state the completeness property of $\mathbb{R}$. Throughout this course, we assume this is true.

Axiom (Completeness property of $\mathbb{R}$). *Every non-empty subset of $\mathbb{R}$ which is bounded above has the least upper bound.*

The above axiom is also stated as that $\mathbb{R}$ has the *least upper bound property*.

Theorem 1.2. *$\mathbb{R}$ has the **greatest lower bound property** in the sense that every non-empty subset of $\mathbb{R}$ which is bounded below has the greatest lower bound.*

Proof. Suppose E is a non-empty subset of $\mathbb{R}$ which is bounded below. Consider the set $-E = \{-x : x \in E\}$. Then $-E$ is non-empty and bounded above. So $\sup(-E)$ exists. Using the above **Question** we get that $\sup(-E) = -\inf E$, so $\inf E$ exists. □

A consequence of the completeness property is the following theorem.

Theorem 1.3 (Archimedean principle). *For any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $x < n$. In particular, for any $x > 0$ and $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $y < nx$.*

Proof. The desired inequality in the second statement is equivalent to that $y/x < n$, so it follows from the first statement. So it suffices to prove the first statement.

We prove the first statement by contradiction. Assume for a certain $x \in \mathbb{R}$, we have $n \leqslant x$ for all $n \in \mathbb{N}$. This implies x is an upper bound of $\mathbb{N}$. By the least upper bound property of $\mathbb{R}$, $\alpha = \sup \mathbb{N}$ exists. In particular, α is an upper bound of $\mathbb{N}$.

Since $\alpha - 1 < \sup \mathbb{N}$, by Proposition 3 there is $m \in \mathbb{N}$ such that

$$\alpha - 1 < m \leqslant \sup \mathbb{N}.$$

The left inequality implies $\alpha < m + 1$. But $m + 1 \in \mathbb{N}$ so α is not an upper bound of $\mathbb{N}$, which is a contradiction. □

Example 1.6. Find the supremum of the set $E = \left\{ \dfrac{3n}{n+4} : n \in \mathbb{N} \right\}$.

Solution. First note that, $\frac{3n}{n+4} = 3(\frac{n}{n+4}) < 3$. From calculus, we have $\frac{3n}{n+4} \to 3$ as $n \to \infty$. So we are suggested that

$$\sup E = 3.$$

By above we see that 3 is an upper bound of E. Let $r < 3$. We show that r is not an upper bound. This means $\exists\, n \in \mathbb{N}$ such that

$$r < \frac{3n}{n+4} \quad \Leftrightarrow \quad \frac{4r}{3-r} < n.$$

We apply the Archimedean principle with $x = \frac{4r}{3-r}$ to get such an n.

Now every r such that $r < 3$ is not an upper bound of E, so we conclude that $\sup E = 3$ as desired. ∎

The following fact is known as the *well-ordering principle.*

Question. Show that every non-empty subset of $\mathbb{N}$ has the smallest element. Similarly, show that every non-empty subset of $\mathbb{N}$ which is bounded above has the largest element.

Corollary 1.1. *For any $x \in \mathbb{R}$, there is $n \in \mathbb{Z}$ such that*

$$n \leqslant x < n+1.$$

Proof. By the Archimedean property, $\exists\, N \in \mathbb{N}$ such that $-x < N$. So $-N < x$. Considering the set $E = \{n \in \mathbb{Z} : n \leqslant x\}$. Since $-N \in E$, E is non-empty and is bounded above (by x). Then $n := \sup E$ exists in $\mathbb{R}$ and n is an integer.[1] Is is clear that $n + 1 > x$. Therefore $n \leqslant x < n + 1$. □

The integer n according to the above corollary is the *floor function* at x, denoted by $\lfloor x \rfloor$. So we have

$$\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1.$$

Employing the floor function, we can prove the *density property* of $\mathbb{Q}$ in $\mathbb{R}$.

Corollary 1.2. *Let $x, y \in \mathbb{R}$ and $x < y$. Then $\exists\, r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. By the Archimedean property, $\exists\, n \in \mathbb{N}$ such that $1/(y - x) < n$. So

$$n(y-x) > 1 \quad \Rightarrow \quad ny > nx + 1.$$

Let $m = \lfloor nx \rfloor + 1 \in \mathbb{Z}$. Then $m \leqslant nx + 1 < ny$ and $m > (nx - 1) + 1 = nx$, that is

$$nx < m < ny \quad \Rightarrow \quad x < \frac{m}{n} < y.$$

□

Example 1.7. Let $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Find $\sup E$ and $\inf E$.

Solution. We note that $0 < \frac{1}{n} < 1$ for all $n \in \mathbb{N}$. So 0 is a lower bound of E and 1 is an upper bound of E.

[1] This is a consequence of the well-ordering principle.

Let $r > 0$ be a real number. By the Archimedean principle, we can choose $n \in \mathbb{N}$ such that $n > 1/r$. Then $r > 1/n$ which implies that r is not a lower bound of E. We conclude that $0 = \inf E$.

The set of upper bounds of E is $[1, \infty)$, so $\sup E = 1$. ■

Example 1.8. Let $E = \left\{ \dfrac{n+1}{n} : n \in \mathbb{N} \right\}$. Find $\sup E, \inf E$.

Solution. Note that $(n+1)/n = 1 + 1/n$, so $1 \leqslant x \leqslant 2$ for all $x \in E$. So 1 is a lower bound of E and 2 is an upper bound of E. The set of upper bounds of E is $[2, \infty)$, hence $\sup E = 2$.

Let $l = \inf E$. Clearly $l \geqslant 1$. Then $l \leqslant (n+1)/n$ for all $n \in \mathbb{N}$, hence $l - 1 \leqslant 1/n$ for all $n \in \mathbb{N}$. It follows from the following exercise that $l - 1 \leqslant 0$. Therefore $l = 1$. ■

Another consequence of the completeness property is the following result.

Theorem 1.4 (**Nested interval theorem**). *Let $I_n = [a_n, b_n]$ $(n \in \mathbb{N})$ be closed intervals where $a_n \leqslant b_n$ and $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.*

Proof. We have show that $\exists\, c \in I_n$ for all n. Define $E = \{a_n : n \in \mathbb{N}\}$. Since $I_{n+1} \subset I_n$ for all n, we have

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \quad \text{and} \quad b_1 \geqslant b_2 \geqslant b_3 \geqslant \cdots .$$

Claim. For all $n, m \in \mathbb{N}$, we have $a_m \leqslant b_n$.

Proof of Claim. If $n \leqslant m$, then $a_m \leqslant b_m$ and $b_m \leqslant b_n$. Hence $a_m \leqslant b_n$ is true. If $n > m$ then $a_m \leqslant a_n$ and $a_n \leqslant b_n$. Hence $a_m \leqslant b_n$ is true. So the claim is proved.

The claim implies that every b_n is an upper bound of E. In particular, $E \neq \emptyset$ is bounded above. By the completeness property of $\mathbb{R}$, $c = \sup E$ exists. Since c is the least upper bound of E and every b_n is an upper bound of E, we get $c \leqslant b_n$ for all $n \in \mathbb{N}$. Since c is an upper bound of E, we have $a_n \leqslant c$ for all n. Therefore $c \in I_n$ for all n as desired. □

Question. Is the result of the nested interval theorem still valid when closed intervals are changed into open intervals $I_n = (a_n, b_n)$?

Exercise 1.3.

1. *Let $E = \{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}$. Find $\sup E, \inf E$.*

2. *Let $E \subset \mathbb{R}$ be a non-empty bounded set of positive real numbers and $F = \{x^2 : x \in E\}$. If $u = \sup E$, prove that $\sup F = u^2$. Give a counter-example in the case that elements of E can be negative numbers.*

3. *Let A, B be non-empty, bounded subsets of $\mathbb{R}$ and $c \in \mathbb{R}$. Define $c + A = \{c + x : x \in A\}$, $cA = \{cx : x \in A\}$, and $A + B = \{x + y : x \in A, y \in B\}$.*

Prove that

$$\sup(c+A) = c + \sup A, \quad \inf(c+A) = c + \inf A,$$
$$\sup(cA) = \begin{cases} c\sup A & c \geqslant 0, \\ c\inf A & c < 0, \end{cases} \quad \inf(cA) = \begin{cases} c\inf A & c \geqslant 0, \\ c\sup A & c < 0, \end{cases}$$
$$\sup(A+B) = \sup A + \sup B \quad \inf(A+B) = \inf A + \inf B.$$

4. *Prove that* $\bigcap_{n=1}^{\infty}(n,\infty) = \emptyset$.

5. *Find the supremum and infimum for each of the following sets.*

   (a) $\left\{(-1)^n \dfrac{3n+2}{2n+3} : n \in \mathbb{N}\right\}$

   (b) $\{x \in \mathbb{R} : x - 2 < 1/(x-1)\}$

   (c) $\{x \in \mathbb{R} : |x-1| + |x-2| \leqslant 3\}$

   (d) $\{x \in \mathbb{R} : \sqrt{x - 1/8} > x\}$

   (e) $\left\{(-1)^n \left(\sin \dfrac{n\pi}{2} - \dfrac{1}{n}\right) : n \in \mathbb{N}\right\}$

Mathematical Analysis: Lecture 3

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we introduce*

1. *$\mathbb{R}$ and $\mathbb{R}^n$ as metric spaces,*
2. *abstract definition of metric spaces,*
3. *basic constructions.*

2 Metric Spaces

In this chapter we introduce the concept of a (non-empty) set X endowed with a "distance" function $d = d(x, y)$ so that we can do analysis on such a set. We shall first recall such a structure on the real number system $\mathbb{R}$ and generally $\mathbb{R}^n$, the Euclidean spaces. Then we define the abstract concept of *metric space*. Many examples and some basic properties of metric spaces will be given. We postpone important analysis contents to latter sections. Then we study the concepts of *connectedness* and *compactness*.

2.1 $\mathbb{R}$ as a metric space

On $\mathbb{R}$, the function

$$d(x, y) = |x - y|$$

plays the role as distance between x and y. As will be recalled, many concepts in analysis can be expressed in terms of d. For example, the continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at $a \in \mathbb{R}$ can be expressed as

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that} \quad (d(x, a) < \delta) \Rightarrow (d(f(x), f(a)) < \varepsilon).$$

Lemma 2.1. *On $\mathbb{R}$, the function $d(x, y) = |x - y|$ satisfies the following identities*

(1) If $x \neq y$ then $d(x, y) > 0$ and $d(x, x) = 0$,

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leqslant d(x,y) + d(y,z)$. ***(The triangle inequality)***

Proof. (1) and (2) are obvious. (3) is the same as $|x-z| \leqslant |x-y| + |y-z|$, which is Exercise 1.2. □

Example 2.1. On $\mathbb{R}$, let

$$\begin{aligned} D_1(x,y) &= (x-y)^2 \\ D_2(x,y) &= \sqrt{|x-y|} \\ D_3(x,y) &= |x^2 - y^2|. \end{aligned}$$

Determine whether D_1, D_2, D_3 satisfies all properties (1)-(3) of the preceding lemma.

Solution. By setting $y = -x$, we have $D_3(x,-x) = 0$ for all $x \in \mathbb{R}$, so D_3 fails to satisfy the property (1).

For D_1 and D_2, the properties (1)-(2) in the preceding lemma is obviously true. Consider

$$D_1(x,z) = (x-z)^2 = (x-y)^2 + 2(x-y)(y-z) + (y-z)^2 > D_1(x,y) + D_1(y,z)$$

provided $(x-y)(y-z) > 0$. We can take for example, $x - y = 2, y - z = 3$. Thus D_1 fails to satisfies (3) of the lemma.

We show that D_2 satisfies (3) of the lemma. For $x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} D_2(x,z)^2 &= |x-z| \\ &\leqslant |x-y| + |y-z| \\ &= D_2(x,y)^2 + D_2(y,z)^2 \\ &\leqslant (D_2(x,y) + D_2(y,z))^2, \end{aligned}$$

where we have used that $D_2(x,y), D_2(y,z) \geqslant 0$. Thus D_2 satisfies (3). ■

Lemma 2.2. *In $\mathbb{R}$, the triangle inequality is equivalent to that*

$$|d(x,z) - d(y,z)| \leqslant d(x,y) \quad \textit{for all } x, y, z \in \mathbb{R}.$$

Proof. The inequality is equivalent to that

$$||x-z| - |y-z|| \leqslant |x-y|,$$

which follows from Proposition 2. □

2.2 The Euclidean spaces $\mathbb{R}^n$

From calculus, we already familiar with functions of two variables $f(x,y)$ and three variables $g(x,y,z)$. They are defined as functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$, respectively. One can generalize this to function in n-variables $f(x_1, x_2, \ldots, x_n)$ and defines it on the set $\mathbb{R}^n$ of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$.

Definition 2.1. Let $n \in \mathbb{N}$. We define the Cartesian product

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

Elements of $\mathbb{R}^n$ are denoted by

$$\vec{x}, \quad \vec{y}, \quad \vec{z},$$

and they are called **vectors**. Real numbers will be called **scalar**.

We define on $\mathbb{R}^n$ the *vector addition* and the *scalar multiplication* operations:

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$
$$c\vec{x} = (cx_1, cx_2, \ldots, cx_n),$$

so that $\mathbb{R}^n$ becomes a vector space according to the following theorem.

Theorem 2.1 ($\mathbb{R}^n$ **as a vector space**)**.** *For each $n \in \mathbb{N}$, $\mathbb{R}^n$ with the above operations is a vector space according to the following properties*

(A1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

(A2) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

(A3) $\exists\, \vec{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$ *such that* $\vec{0} + \vec{x} = \vec{x}$ *for all* $\vec{x} \in \mathbb{R}^n$.

(A4) $\forall\, \vec{x} \in \mathbb{R}^n, \exists\, (-\vec{x}) \in \mathbb{R}^n$ *such that* $\vec{x} + (-\vec{x}) = 0$.

(S1) $1\vec{x} = \vec{x}$

(S2) $b(c\vec{x}) = (bc)\vec{x}$

(D) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$ *and* $(b + c)\vec{x} = b\vec{x} + c\vec{x}$.

Proof. Straightforward. □

Remark. For the definition of metric space given below, it does not require the above vector space structure. It contains merely a *distance structure*.

We define the dot product, the norm, and the standard Euclidean distance on $\mathbb{R}^n$ as follows.

Definition 2.2. We define the **scalar** (or **dot product**) in $\mathbb{R}^n$ by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

the **Euclidean norm** by

$$\|x\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and, finally, the **Euclidean metric** (or **Euclidean distance**) by

$$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}.$$

Question. Show that

$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 - 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2,$$
$$\|\vec{x}\| \geqslant |x_i|, \quad \text{for all } i = 1, 2, \ldots, n,$$

and

$$(a\vec{x})\cdot\vec{y} = a(\vec{x}\cdot\vec{y}) = \vec{x}\cdot(a\vec{y}).$$

Question. If a quadratic polynomial $at^2 + bt + c \geqslant 0$ for all $t \in \mathbb{R}$ and $a \geqslant 0$, show that $b^2 \leqslant 4ac$.

Lemma 2.3. *We have*

(1) $|\vec{x}\cdot\vec{y}| \leqslant \|\vec{x}\|\|\vec{y}\|$ ***(Cauchy-Schwarz inequality)***

(2) d satisfies properties (1)-(3) of Lemma 2.1.

Proof. (1) Let $p(t) = \|t\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2t^2 - 2(\vec{x}\cdot\vec{y})t + \|\vec{y}\|^2$. Since the quadratic polynomial $p(t) \geqslant 0$ for all $t \in \mathbb{R}$, we have

$$(2\vec{x}\cdot\vec{y})^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \leqslant 0.$$

That is $|\vec{x}\cdot\vec{y}| \leqslant \|\vec{x}\|\|\vec{y}\|$ as claim.

If $\vec{x} \neq \vec{y}$ then $x_i \neq y_i$ for some i hence $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| \geqslant |x_i - y_i| > 0$. It is obvious that $d(\vec{x},\vec{x}) = 0$ and $d(\vec{x},\vec{y}) = d(\vec{y},\vec{x})$.

The triangle inequality is equivalent to $\|\vec{x}-\vec{z}\| \leqslant \|\vec{x}-\vec{y}\| + \|\vec{y}-\vec{z}\|$ for all $\vec{x},\vec{y},\vec{z} \in \mathbb{R}^n$ and it will be left to the reader (see Exercise 2.1). □

2.3 The definition of metric spaces and examples

Now we introduce the abstract definition of metric space.

Definition 2.3. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called **metric** on X if it satisfies

(1) If x, y are distinct elements of X then $d(x,y) > 0$, and $d(x,x) = 0$.

(2) $d(x,y) = d(y,x)$

(3) $d(x, z) \leqslant d(x, y) + d(y, z)$ (**The triangle inequality**)

If d is a metric on X, the pair (X, d) is called a **metric space**.

Example 2.2. We have already shown that $\mathbb{R}$ with the absolute value function

$$d(x, y) = |x - y| \quad (x, y \in \mathbb{R})$$

is a metric space. It is called the *standard metric* on $\mathbb{R}$.

For any $n \in \mathbb{N}$, it was shown in Lemma 2.3 that $\mathbb{R}^n$ endowed with the Euclidean distance

$$d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \quad (\vec{x}, \vec{y} \in \mathbb{R}^n)$$

is a metric space. It is called the *Euclidean metric.* ∎

On arbitrary non-empty set, one can always introduce the following metric.

Example 2.3 (Discrete metric)**.** Let X be a non-empty set. Define $\delta : X \times X \to \mathbb{R}$ by

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Show that (X, δ) is a metric space. It is called a *discrete metric space.*

Solution. Clearly, we have

$$\delta(x, y) = 1 > 0 \quad \text{if } x \neq y, \quad \delta(x, x) = 0.$$

Also, $\delta(x, y) = \delta(y, x)$. For the triangle inequality, if $x = z$ then obviously, $\delta(x, z) = 0 \leqslant \delta(x, y) + \delta(y, z)$. On the other hand, if $x \neq z$ then $x \neq y$ or otherwise $y \neq z$. This implies

$$\delta(x, z) = 1 \leqslant \delta(x, y) + \delta(y, z),$$

because at least one in $\delta(x, y), \delta(y, z)$ is equal to 1. Therefore δ is a metric. ∎

Example 2.4 (Taxicab metric)**.** On $\mathbb{R}^2$, define

$$d_1(\vec{x}, \vec{y}) = |x_1 - y_2| + |x_2 - y_2|.$$

Show that d_1 is a metric on $\mathbb{R}^2$. It is called the *taxicab metric*, see Exercise 2.1.

Solution. If $\vec{x} \neq \vec{y}$ then $x_1 - y_1 \neq 0$ or $x_2 - y_2 \neq 0$. This implies

$$d_2(\vec{x}, \vec{y}) > 0.$$

Also, it is clear that $d_2(\vec{x}, \vec{x}) = 0$ and $d_2(\vec{x}, \vec{y}) = d_2(\vec{y}, \vec{x})$.

We have to verify the triangle inequality. We have

$$\begin{aligned} d_2(\vec{x}, \vec{z}) &= |x_1 - z_1| + |x_2 - z_2| \\ &\leqslant (|x_1 - y_1| + |y_1 - z_1|) + (|x_2 - y_2| + |y_2 - z_2|) \\ &= d_2(\vec{x}, \vec{y}) + d_2(\vec{y}, \vec{z}) \end{aligned}$$

which means d_2 satisfies the triangle inequality. ∎

Question. On $\mathbb{R}^2$, define

$$D(\vec{x}, \vec{y}) = |x_1 - y_1| \quad (\vec{x}, \vec{y} \in \mathbb{R}^2).$$

Is $(\mathbb{R}^2, D)$ a metric space?

Lemma 2.4. *The triangle inequality is equivalent to that*

$$|d(x,y) - d(x,z)| \leqslant d(y,z) \quad \textit{for all } x, y, z \in X.$$

Proof. $(\Rightarrow)$ Suppose the triangle inequality is true, i.e. $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in X$. Then

$$d(x,z) - d(x,y) \leqslant d(y,z) \quad \text{for all } x, y, z \in X.$$

Changing the role of y, z, we also get

$$d(x,y) - d(x,z) \leqslant d(z,y).$$

But $d(y,z) = d(z,y)$, so we get $|d(x,y) - d(x,z)| \leqslant d(y,z)$.

$(\Leftarrow)$ Obvious. □

Next, we present three basic construction of metric spaces

1. metric subspaces,
2. pullbacks,
3. product metrics.

Theorem 2.2 (Metric subspaces). *If (X, d) is a metric space then for every non-emptyset $E \subset X$, (E, d) is a metric space. It is called a* ***metric subspace****.*

Proof. See **Question** below. □

Question. Prove the assertion in Theorem 2.2.

Example 2.5. The set $X = (0, \infty)$ with the absolute value

$$d(x,y) = |x - y| \quad (x, y > 0)$$

is a metric space. It is a metric subspace of $\mathbb{R}$.

The quarter plane $X = \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0\}$ equipped with the taxicab metric

$$d_1(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$$

is a metric subspace of $\mathbb{R}^2$. ■

Lemma 2.5 (Pullback of metric). *Let $X \neq \emptyset$ be a non-empty set and (Y, ρ) a metric space. Suppose $f : X \to Y$ is a one-to-one map. Then*

$$d(x, y) := \rho(f(x), f(y)) \quad (x, y \in X)$$

defines a metric on X so that (X, d) is a metric space.

Proof. Since ρ is a metric on Y, ρ satisfies the properties (1)-(3) of metric space. If $x \neq y$, since f is one-to-one, we get $f(x) \neq f(y)$; hence $\rho(f(x), f(y)) > 0$. This means

$$d(x, y) = \rho(f(x), f(y)) > 0.$$

Also, we have

$$d(x, x) = \rho(f(x), f(x)) = 0$$

and

$$d(x, y) = \rho(f(x), f(y)) = \rho(f(y), f(x)) = d(y, x).$$

Consider

$$\begin{aligned} d(x, z) &= \rho(f(x), f(z)) \\ &\leqslant \rho(f(x), f(y)) + \rho(f(y), f(z)) \quad \text{(triangle inequality for } \rho) \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus d is a metric on X. □

Example 2.6. On $X = (0, \infty)$, prove that

$$d(x, y) = \left| \ln \frac{x}{y} \right| \quad (x, y > 0)$$

defines a metric.

Solution. Let $Y = \mathbb{R}$ with the metric $\rho(u, v) = |u - v|$. Consider the map $f : X \to Y$ given by $f(x) = \ln x$ $(x > 0)$. Clearly, f is one-to-one. Hence

$$d(x, y) = \rho(f(x), f(y)) = |\ln x - \ln y| = \left| \ln \frac{x}{y} \right|$$

is a metric on X by Lemma 2.5. ■

Question. Let $k > 0$ be a constant. Show that $d(x, y) = k|x - y| (x, y \in \mathbb{R})$ is a metric on $\mathbb{R}$.

Question. Show that $d(x, y) = |\sqrt[3]{x} - \sqrt[3]{y}|$ is a metric on $\mathbb{R}$.

Definition 2.4. Let $(X, d), (Y, \rho)$ be metric spaces. Define ψ on $X \times Y$ by

$$\psi((x, \xi), (y, \eta)) = d(x, y) + \rho(\xi, \eta).$$

ψ is called the **product metric** and is often denoted by $\psi = d \times \rho$.

Theorem 2.3 (Product metric spaces). *$(X \times Y, d \times \rho)$ is a metric space.*

Proof. Let $\psi = d \times \rho$. If $(x, \xi) \neq (y, \eta)$ we have either $x \neq y$ or $\xi \neq \eta$. Thus $d(x, y) > 0$ or $\rho(\xi, \eta) > 0$, hence $\psi((x, \xi), (y, \eta)) > 0$. Clearly, $\psi((x, \xi), (x, \xi)) = 0$ and $\psi((x, \xi), (y, \eta)) = \psi((y, \eta), (x, \xi))$.

Let us prove the triangle inequality. Consider

$$\begin{aligned} \psi((x, \xi), (z, \tau)) &= d(x, z) + \rho(\xi, \tau) \\ &\leqslant [d(x, y) + d(y, z)] + [\rho(\xi, \eta) + \rho(\eta, \tau)] \\ &= \psi((x, \xi), (y, \eta)) + \psi((y, \eta), (z, \tau)), \end{aligned}$$

so the triangle inequality is true for ψ. □

Example 2.7. Show that

$$D(\vec{x}, \vec{y}) = |x_1 - y_1| + 3|x_2 - y_2| \quad (\vec{x}, \vec{y} \in \mathbb{R}^2)$$

is a metric on $\mathbb{R}^2$.

Solution. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Let $d(x, y) = |x - y|$ and $\rho(z, w) = 3|z - w|$. Then d is a metric on $\mathbb{R}$. That ρ is a metric on $\mathbb{R}$ is true by pullback $f(x) = 3x$. ■

Question. Show that on $\mathbb{R}^3_+ = \{\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$,

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} + \left| \ln \frac{x_3}{y_3} \right|$$

is a metric on $\mathbb{R}^3_+$.

Exercise 2.1.

1. *For each of the following functions on $\mathbb{R}^2$, determine whether it is a metric.*

   (a) $D_1(\vec{x}, \vec{y}) = |e^{x_1} - e^{x_2}| + |e^{y_1} - e^{y_2}|$

   (b) $D_2(\vec{x}, \vec{y}) = \dfrac{|x_1 - y_1| + |x_2 - y_2|}{2 + |x_1 - y_1| + |x_2 - y_2|}$

2. *Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$.*

   Prove that $\|\vec{x} - \vec{z}\| \leqslant \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$.

3. *(Taxicab and supremum metrics). On $\mathbb{R}^n$, we define*

$$\|\vec{x}\|_1 = \sum_{j=1}^{n} |x_j|, \quad \|\vec{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

Set $d_1(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|_1$ *and* $d_\infty(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|_\infty$.

Prove that d_1, d_∞ *are metrics on* $\mathbb{R}^n$.

4. *Let* (X,d) *be a metric space. Prove that*

$$|d(x,y)-d(z,w)| \leqslant d(x,z)+d(y,w) \quad (\forall\, x,y,z,w \in X).$$

5. *Let* (X,d) *be a metric space and* $M>0$ *a constant. Define*

$$\rho(x,y) = \min\{d(x,y), M\}.$$

Prove that ρ *is a metric on* X.

6. *Let* d,ρ *be metrics on a non-empty set* X. *Define* $\gamma : X\times X \to \mathbb{R}$ *by*

$$\gamma(x,y) = \max\{d(x,y), \rho(x,y)\} \quad (x,y\in X).$$

Prove that γ *is a metric on* X.

Mathematical Analysis: Lecture 4

Sujin Khomrutai, Ph.D.

Summary. *The aim of this lecture is to introduce the following*

1. *open balls and closed balls,*
2. *open sets and closed sets,*
3. *basic properties of open and closed sets.*

2.4 Metric topology

As we will learn throughout the course, fundamental concepts in analysis on a metric space such as continuity, limits, etc. can be defined from the metric. In the process, it will be shown however that these concepts can be rephrased in terms of the special subsets known as"*open subsets*". The aim of this section is to introduce such notion.

Definition 2.5. Let (X, d) be a metric space, $x_0 \in X$, and $r > 0$. We denote

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$

It is called the **open ball with center x_0 and radius r.**

We also denote

$$\bar{B}_r(x_0) = \{x \in X : d(x, x_0) \leqslant r\}$$

It is called the **closed ball with center x_0 and radius r.**

Question. Why $x_0 \in B_r(x_0)$ and $B_r(x_0) \subset \bar{B}_r(x_0)$?

Remark. When there are more than one metrics on the same set, we will use the superscript to distinguish open balls, e.g. if d, ρ are metrics on X, we denote

$$B_r^d(x_0), \quad B_r^\rho(x_0)$$

for open balls with respect to d and ρ, respectively.

We present some examples.

Example 2.8. Let X be a non-empty set equipped with the discrete metric. See Example 2.3.

For $r \in (0,1)$ and $x_0 \in X$, we have

$$B_r(x_0) = \bar{B}_r(x_0) = \{x_0\},$$

whereas, for $r = 1$,

$$B_1(x_0) = \{x_0\}, \quad \bar{B}_r(x_0) = X.$$

In particular, the collection of all open balls is $\{X\} \cup \bigcup_{x \in X} \{\{x\}\}$. ■

Note. From now on, $\mathbb{R}$ has the standard metric and $\mathbb{R}^n$ has the Euclidean metric, unless otherwise specified.

Example 2.9. In $\mathbb{R}$, open balls are open intervals

$$B_r(x_0) = (x_0 - r, x_0 + r),$$

and closed balls are closed intervals

$$\bar{B}_r(x_0) = [x_0 - r, x_0 + r].$$

In particular, the collection of open balls is $\{(a,b) : a < b\}$. ■

Question. What are open balls in $\mathbb{R}^2$ and $\mathbb{R}^3$?

Example 2.10. According to Exercise 2.1, $\mathbb{R}^2$ with the map d_∞ defined by

$$d_\infty(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

is a metric space. Find all open balls $B_r^{d_\infty}(\vec{x})$.

Solution. We have $\vec{y} \in B_r^{d_\infty}(\vec{x})$ precisely when

$$\max\{|y_1 - x_1|, |y_2 - x_2|\} < r \quad \Leftrightarrow \quad |x_1 - y_1| < r \text{ and } |x_2 - y_2| < r.$$

Thus the open ball $B_r^{d_\infty}(\vec{x}) = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$, which is a square centered at $\vec{x}$ and side parallel to the x- and y-axes with length $2r$. ■

Now we introduce the basic topological concept.

Definition 2.6. Let (X, d) be a metric space.

(1) A subset $U \subset X$ is called an **open set** provided

$$\forall\, x_0 \in U, \exists\, r > 0 \quad \text{such that} \quad B_r(x_0) \subset U.$$

The collection $\mathcal{T}$ of all open subsets of X is called the **metric topology** of X. When there are at least two spaces, we will denote the topology by $\mathcal{T}_X$.

(2) A subset $G \subset X$ is called a **closed set** provided its complement

$$G^c := X \setminus G \quad \text{is an open set.}$$

Question. Explain why $\emptyset$ and X are open sets?

Question. Explain why $\emptyset$ and X are closed sets?

Remark. As will be shown, being open or closed sets are not negate properties of each other. For example, in $\mathbb{R}$ the set $(0, 1]$ is neither open nor closed.

Example 2.11 (Discrete topology)**.** Let X be a non-empty set which is equipped with the discrete metric δ. See Example 2.3.

Show that the metric topology is

$$\mathcal{P}(X) = \text{the collection of all subsets of } X.$$

It is called the **discrete topology** on X.

Solution. Let $E \subset X$. If $E = \emptyset$, then E is open by the **Question** above. Suppose E is non-empty.

For any $x \in E$, we have $B_{1/2}(x_0) = \{x_0\} \subset E$. So E is an open set. Therefore, the metric topology in this case is the power set of X. ∎

Employing the *triangle inequality*, we get the following results.

Lemma 2.6 (**Every open ball is an open set**)**.** *In a metric space (X, d), every ball $B_r(x_0)$ is an open set.*

Proof. Let $y \in B_r(x_0)$ and $l = d(x_0, y)$. Since $y \in B_r(x_0)$, we have $d(x_0, y) < r$, that is $t := r - l > 0$. We claim that

$$B_t(y) \subset B_r(x_0).$$

For all $z \in B_t(y)$ we have $d(y, z) < t$. So by the triangle inequality we get

$$d(x_0, z) \leqslant d(x_0, y) + d(y, z) < l + t = r,$$

which means $z \in B_r(x_0)$. Therefore the claim is true, hence $B_r(x_0)$ is an open set. □

Example 2.12. In $\mathbb{R}$, prove that any open interval (a, b) is an open set.

Solution. Since

$$(a, b) = B_r(x_0), \quad \text{where } r = (b - a)/2, x_0 = (a + b)/2,$$

the conclusion is true by the preceding lemma. ∎

Lemma 2.7 (**Every closed ball is a closed set**)**.** *In a metric space (X, d), every closed ball $\bar{B}_r(x_0)$ is a closed set.*

Proof. Let $U = X \setminus \bar{B}_r(x_0) = \{x \in X : d(x, x_0) > r\}$. We have to show that U is an open set, i.e.

$$\forall\, z_0 \in U, \exists\, R > 0 \quad \text{such that} \quad B_R(z_0) \subset U.$$

Let $z_0 \in U$. Set $R = d(z_0, x_0) - r$. Then $R > 0$.

Consider $y \in B_R(z_0)$. We have $d(y, z_0) < R$. By Lemma 2.4, we get

$$d(x_0, y) \geqslant d(x_0, z_0) - d(y, z_0) > d(x_0, z_0) - R = r.$$

Thus $y \in U$, and this is true for all $y \in B_R(z_0)$, hence $B_R(z_0) \subset U$. □

Theorem 2.4. *Let (X, d) be a metric space. The metric topology $\mathcal{T}$ of X satisfies the following properties.*

(T0) $\emptyset, X \in \mathcal{T}$.

*(T1) (**Arbitrary unions**) If $\{U_\alpha\}_{\alpha \in A}$ is a family of elements in $\mathcal{T}$, then*

$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}.$$

*(T2) (**Finite intersections**) If $U_1, U_2, \ldots, U_N$ is a finite family of elements in $\mathcal{T}$, then*

$$\bigcap_{k=1}^{N} U_k \in \mathcal{T}.$$

Proof. (T0) Clearly, X is an open set. By the preceding remark, $\emptyset$ is also open.

Since $\emptyset = X^c$ and $X = \emptyset^c$, we conclude that $\emptyset$ and X are also closed sets.

(T1) Let $U = \bigcup_{\alpha \in A} U_\alpha$. If $x_0 \in U$ then $\exists\, \alpha \in A$ such that $x_0 \in U_\alpha$. Since U_α is an open set, there is $r > 0$ such that

$$B_r(x_0) \subset U_\alpha.$$

Thus

$$B_r(x_0) \subset U_\alpha \subset U,$$

This is true for all $x_0 \in U$, so we conclude that U is open.

(T2) Let $U = \bigcap_{k=1}^{N} U_k$ and $x_0 \in U$. Since U_k are open sets, there are $r_k > 0$ such that $B_{r_k}(x_0) \subset U_k$ for $k = 1, 2, \ldots, N$. Let

$$r = \min\{r_1, \ldots, r_N\}.$$

Then

$$B_r(x_0) \subset B_{r_k}(x_0) \subset U_k \quad \text{for all } k = 1, 2, \ldots, N,$$

hence $B_r(x_0) \subset \bigcap_{k=1}^{N} U_k = U$. Therefore U is an open set. □

Remark. If there are infinitely many open sets, their intersection may not be an open set. For example, on $\mathbb{R}$, we have

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}$$

and $\{0\}$ is a closed set by Lemma 2.7.

Question. In $\mathbb{R}$, show that (a,∞) and $(-\infty,a)$ are open sets.

Example 2.13. Show that every singleton set $\{a\}$ in $\mathbb{R}$ is a closed set.

Solution. We have $\mathbb{R}\setminus\{a\}=(-\infty,a)\cup(a,\infty)$. Now $(-\infty,a)\cup(a,\infty)$ are open sets. So $\{a\}$ is a closed set. ■

Example 2.14. Every open interval (a,b) in $\mathbb{R}$ is an open set.

Every closed interval $[a,b]$ in $\mathbb{R}$ is a closed set.

A union of open intervals $\bigcup_{\alpha\in A}(a_\alpha,b_\alpha)$ is an open set.

A finite intersection of open intervals $(a_1,b_1)\cap\cdots\cap(a_k,b_k)$ is an open set. ■

Exercise 2.2.

1. *In $\mathbb{R}^2$ with the metric $d_1(\vec{x},\vec{y})=|x_1-y_1|+|x_2-y_2|$, describe a ball $B_r(\vec{x}_0)$.*
2. *Determine whether $\{\frac{1}{n}:n\in\mathbb{N}\}$ is open or closed in $\mathbb{R}$.*
3. *Let x,y be distinct elements in a metric space (X,d). Find the largest number $r>0$ such that $B_r(x)\cap B_r(y)=\emptyset$.*
4. *In a metric space (X,d), show that $\{x\in X:d(x,x_0)=r\}$ where $x_0\in X,r\geqslant 0$, is a closed set.*
5. *Find two closed subsets A,B of $\mathbb{R}$ (standard metric) which are non-empty and such that $A\cap B=\emptyset$ and $\inf\{|x-y|:x\in A,y\in B\}=0$.*
6. *Prove that in a metric space, arbitrary intersection of closed sets and a finite union of closed sets are closed sets.*

■

Mathematical Analysis: Lecture 5

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we discuss*

1. *(Metric) subspace topology*
2. *Connectedness.*
3. *$\mathbb{R}$ and $\mathbb{R}^n$ are connected.*
4. *The connected subsets of $\mathbb{R}$ are intervals.*

2.5 (Metric) subspaces topology

Throughout this note, let (X, d) be a metric space. Let $\mathcal{T}$ denote the metric topology on X, that is

$$\mathcal{T} = \text{the collection of all open subsets of } X.$$

Consider a non-empty subset $E \subset X$. By Theorem 2.2, E has an induced metric space structure given by the metric still denoted by d. We will study the induced metric topology $\mathcal{T}_E$ on E. Balls in E and X will be denoted, respectively by

$$B_r^E(x_0), \quad B_r(x_0).$$

Question. If $A = A' \cap E$, show that $(X \setminus A') \cap E = E \setminus A$.

Lemma 2.8. *Let (E, d) be a metric subspace of a metric space (X, d).*

*(1) (**Open balls in a metric subspace**) $B_r^E(x_0) = B_r(x_0) \cap E$ for $x_0 \in E, r > 0$.*

*(2) (**Open sets in a metric subspace**) A set $U \subset E$ is an open set in E if and only if $U = U' \cap E$, for an open subset $U' \subset X$.*

*(3) (**Closed sets in a metric subspace**) A set $G \subset E$ is a closed set in E if and only if $G = G' \cap E$, for a closed subset $G' \subset X$.*

Proof. (1) Clearly, $B_r^E(x_0) = \{x \in E : d(x, x_0) < r\} = B_r(x_0) \cap E$.

(2) Let $U \subset E$ be a subset.

($\Leftarrow$) Suppose there is an open subset $U' \subset X$ such that $U = U' \cap E$. Since U' is open in X, we have

$$\forall\, x_0 \in U', \exists\, r > 0 \quad \text{such that} \quad B_r(x_0) \subset U'.$$

Let $x_0 \in U$. Then $x_0 \in U'$ and we have $B_r^E(x_0) = B_r(x_0) \cap E \subset U' \cap E$. This means $B_r^E(x_0) \subset U$, so U is open in E.

($\Rightarrow$) Suppose U is an open subset of E. Then

$$\forall\, x_0 \in U, \exists\, r = r(x_0) > 0 \quad \text{such that} \quad B_r^E(x_0) \subset U.$$

We define

$$U' = \bigcup_{x_0 \in U} B_{r(x_0)}(x_0)$$

By (T1) of Theorem 2.4, U' is an open set in X.

Now $U' \cap E = E \cap \bigcup_{x_0 \in U} B_r(x_0) = \bigcup_{x_0 \in U}(E \cap B_r(x_0)) = \bigcup_{x_0 \in U} B_r^E(x_0) = U$.

(3) ($\Rightarrow$) Suppose G is closed in E. Then $U = E \setminus G$ is open in E. By (2), there is an open subset U' of X such that $U = U' \cap E$. Let $G' = X \setminus U'$. It is a closed subset of X. Furthermore, $G' \cap E = (X \setminus U') \cap E = E \setminus U = G$, by **Question**.

($\Leftarrow$) Suppose $G = G' \cap E$ where G' is closed in X. Then $X \setminus G'$ is open in X, hence $E \setminus G = (X \setminus G') \cap E$ is open in E. Thus G is closed in E. □

From Theorem 2.4 and Lemma 2.8, we define abstract topological spaces and subspaces as follows.

Definition 2.7 (Topological space and subspaces)**.** Let X be a set and $\mathcal{T}$ a collection of subsets of X. Suppose $\mathcal{T}$ satisfies the following properties.

1. $\emptyset, X \in \mathcal{T}$;
2. If $\{U_\alpha\}_{\alpha \in A} \in \mathcal{T}$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$;
3. If $U_1, \dots, U_N \in \mathcal{T}$ then $\bigcap_{i=1}^N U_i \in \mathcal{T}$.

Then $\mathcal{T}$ is called a **topology** on X and $(X, \mathcal{T})$ is called a **topological space**.

Let $E \subset X$, a topological space with topology $\mathcal{T}$. Define

$$\mathcal{T}_E = \{U \cap E : U \in \mathcal{T}\}.$$

Then (see **Question** below) $\mathcal{T}_E$ is a topology on E called the **subspace topology** on E (induced from X). It is also called the **relative topology** on E.

Question. Show that $\mathcal{T}_E$ is a topology on E.

Remark. Consider the following situation. Let X be a topological space and $E \subset X$ has the relative topology from X. When talking about subsets of E, which are also subsets of X, there could be a confusion on the notion of open/closed. To avoid that, we introduce the following notions.

1. A subset of $A \subset E$ is called **relatively open** in E if A is open with respect to the relative topology on E.

2. A subset $B \subset E$ is called **relatively closed** in E if B is closed with respect to the relative topology on E.

3. Simply open/closed will mean with respect to the topology of X.

Question. Show that $(0, 1)$ is open in $\mathbb{R}$, but it is relatively closed as a subset of itslef.

Example 2.15. Show that the relative topology on $\mathbb{N}$ induced from $\mathbb{R}$ is the discrete topology, $\mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N})$.

Solution. We show that every subset $E \subset \mathbb{N}$ is relatively open. Let $U = \bigcup_{k\in E}(k - 1/2, k + 1/2)$. Then U is open in $\mathbb{R}$ and $E = U \cap \mathbb{N}$. So E is relatively open in $\mathbb{N}$. ∎

Example 2.16. Show that

$$E = \{p \in \mathbb{Q} : p > 0, 2 < p^2 < 3\}$$

is both relatively open and closed in $\mathbb{Q}$ (with the relative topology from $\mathbb{R}$).

Solution. Of course $E = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q} = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$. So E is a relatively closed and open in $\mathbb{Q}$, respectively. ∎

Question. For $X = (0, 1) \cup (2, 3)$ with the subspace topology from $\mathbb{R}$, show that $(0, 1)$ and $(2, 3)$ are relatively open and closed in X.

2.6 Connectedness

From (T0) of Theorem 2.4, $\emptyset$ and X are subsets of X that have the property that they are both *open and closed*. An important question arise, is there any other subset E of X having such property?

- For $X = \mathbb{Q}$ with the subspace topology from $\mathbb{R}$, by Example 2.16, the set $E = \{p \in \mathbb{Q} : p > 0, 2 < p^2 < 3\}$ is both open and closed in $\mathbb{Q}$.

- On the other hand, as will be shown, for $X = (0, 1)$, if $E \subset (0, 1)$ is both open and closed, then $E = \emptyset$ or $E = (0, 1)$.

To answer such a question, we introduce a special property called **connectedness** for a metric space.

Definition 2.8. Let (X, d) be a metric space.

1. A pair (A, B) of subsets of X is called a **separation** of X if
   (a) A, B are open sets,
   (b) $A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$,
   (c) $A \cup B = X$.

   We also say that (A, B) **separates** X.
2. If X has no separation, it is called **connected (metric) space**. On the other hand, if X has a separation it is called **disconnected**.
3. For a subset E of X, the notions of separation and connectedness of E are defined by considering the relative topology on E. See **Question** below.

Question. Express the definition of separation for a subspace E of X. Also, what does it mean by $E \subset X$ to be connected?

Theorem 2.5 (**A characterization of connected spaces**). *A metric space X is connected if and only if $\emptyset$ and X are the only sets that are both open and closed.*

In particular, if there is a non-empty set $E \subset X$ such that $E \neq X$ and E is both open and closed, then E is disconnected.

Proof. $(\Rightarrow)$ Suppose X is connected. Assume on the contrary that there is $E \subset X$ such that$E \neq \emptyset, X$ and E is both open and closed. Let $A = E$ and $B = X \setminus E$. It is easy to show that (A, B) separates X. So X is disconnected, a contradiction.

$(\Leftarrow)$ Suppose there is a subset $E \subset X$ such that $E \neq \emptyset, X$ and E is both open and closed. We show that X is disconnected. Same argument as above can be used. □

Example 2.17. Let X be a non-empty set with at least two elements. Endow X with the discrete topology. Let $a \in X$. Define $A = \{a\}$ and $B = X \setminus \{a\}$. Since any subsets of X are open, A, B are open sets. Also, $A \neq \emptyset, B \neq \emptyset$, and $A \cup B = X$. So (A, B) separates X and hence X is disconnected. ■

Example 2.18. By Example 2.16, the set $E = \{p \in \mathbb{Q} : p > 0, 2 < p^2 < 3\}$ is both open and closed in $\mathbb{Q}$. So $\mathbb{Q}$ is disconnected. ■

Question Show that $\mathbb{Q}$ is a disconnected metric space.

By an **interval** in $\mathbb{R}$, we mean a set of one of the following forms

$$(a, b), \quad [a, b), \quad (a, b], \quad [a, b],$$

or

$$(-\infty, b), \quad (-\infty, b], \quad (a, \infty), \quad [a, \infty), \quad (-\infty, \infty),$$

where a, b are finite real numbers such that $a \leqslant b$.

The characterization property of intervals E is that:

$$\text{If } x \in E, y \in E \text{ and } x < z < y \text{ then } z \in E.$$

Theorem 2.6 (**Connected subsets of** $\mathbb{R}$). *A non-empty subset $E \subset \mathbb{R}$ is connected (in the subspace topology) if and only if E is an interval. In particular, $\mathbb{R}$ is a connected metric space.*

Proof. ($\Rightarrow$) We proof the contrapositive (i.e. $\sim q \to \sim p$). Suppose that E is not an interval. This means $\exists\ x, y, z \in \mathbb{R}$ such that $x, y \in E$ and $x < z < y$ and $z \notin E$. Then we are going to show that E is disconnected.

Define

$$A = E \cap (-\infty, z), \quad B = E \cap (z, \infty).$$

Since $x \in A, y \in B$, we have $A \neq \emptyset, B \neq \emptyset$. Clearly, $A \cap B = \emptyset$. Since $(-\infty, z)$ and (z, ∞) are open sets in $\mathbb{R}$, it follows that A, B are relatively open in E. Finally, $A \cup B = E$. Therefore (A, B) separate E, hence E is disconnected.

($\Leftarrow$) Again we prove the contrapositive. Suppose E is disconnected. We have to show that E is not an interval, i.e. $\exists\, x, y, z \in \mathbb{R}$, where $x, y \in E$ and $x < z < y$, such that $z \notin E$.

Since E is disconnected, there is a pair (A, B) which separates E. Let $x \in A$ and $y \in B$, and we can assume $x < y$. Define

$$z = \sup(A \cap [x, y]).$$

Clearly $x \leqslant z \leqslant y$. Since $y \in B = A^c$ and B is relatively open, we have $z \notin B$ and in particular $z < y$.

If $z \notin A$, then we have $x < z < y$ and $z \notin E$. This proves the desired assertion.

If $z \in A$, then by replacing x with z and considering $z_1 = \sup(A \cap [z, y])$, it follows that $z < z_1 < y$ and $z_1 \notin B$. This also proves the desired assertion. $\square$

Example 2.19. Prove that $(0, 1)$ is not a closed set. Moreover, any open interval (a, b) is not a closed set.

Solution. In $\mathbb{R}$, $(0, 1)$ is open. If it were closed, it would implies $\mathbb{R}$ is disconnected, contradicting to the theorem above. ■

Theorem 2.7. *If X, Y are connected metric spaces, then $X \times Y$ is connected.*

Proof. Assume on the contrary that $X \times Y$ is disconnected. So there is a separation (A, B) of $X \times Y$. In particular, A, B are open subsets of $X \times Y$.

Case 1: There is an $x \in X$ such that $(\{x\} \times Y) \cap A$ and $(\{x\} \times Y) \cap B$ are non-empty sets. The projections of $(\{x\} \times Y) \cap A$ and $(\{x\} \times Y) \cap B$ onto Y gives two non-empty open subsets A', B' of Y such that $A' \cap B' = \emptyset$ and $A' \cup B' = Y$. This means Y is disconnected, a contradiction.

Case 2: For any $x \in X$, $\{x\} \times Y$ is either in A or in B. Then the projection of A and B onto X gives non-empty open subsets A', B' of X such that $A' \cap B' = \emptyset$ and $A' \cup B' = X$. This means X is disconnected, a contradiction. □

Question. Prove the converse of the above statement, i.e. If $X \times Y$ is connected, then so are X and Y. (*Hint.* Contrapositive!)

Corollary 2.1. $\mathbb{R}^n$ *is connected.*

Proof. This is true by the preceding theorem. □

Exercise 2.3.

1. *Show that the subset* $E \subset \mathbb{R}^2$ *given by*

$$E = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \geqslant 4\}$$

   is disconnected.

2. *Prove that* $\emptyset$ *is connected.*

3. *Prove that every finite subset having at least two elements of a metric space is disconnected.*

4. *Let* Y *be a metric subspace of a metric space* X. *Prove that a subset* $E \subset Y$ *is a connected subset of* Y *if and only if it is a connected subset of* X.

5. *Show that* $\mathbb{Q}^2$ *(with the subspace topology from* $\mathbb{R}^2$*) is disconnected.*

6. *Prove that* $(0, 1]$ *is neither open nor closed.*

   Hint. Consider connected spaces $(0, \infty)$ *and* $(-\infty, 1]$.

Mathematical Analysis: Lecture 6

Sujin Khomrutai, Ph.D.

Summary. *We study*

1. *Boundedness*
2. *Compactness*
3. *the Heine-Borel property*
4. *the Heine-Borel theorem*

2.7 Compactness

Throughout this note, let (X, d) be a metric space.

Definition 2.9. Let $E \subset X$.

1. An **open cover** of X is a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of X such that
   - all U_α are open sets,
   - $X = \bigcup_{\alpha \in A} U_\alpha$.

   We also say "$\{U_\alpha\}_{\alpha \in A}$ **covers** X" if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X.

2. A **finite sub-cover** of an open cover $\{U_\alpha\}_{\alpha \in A}$ of X is a sub-collection $\{U_{\alpha_i}\}_{i=1}^N$ ($\alpha_i \in A$ for all i) that also covers X.

3. An **open cover** of E is a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of X such that
   - all U_α are open sets,
   - $E \subset \bigcup_{\alpha \in A} U_\alpha$.

A **finite sub-cover** of an open cover $\{U_\alpha\}_{\alpha\in A}$ of E is a sub-collection $\{U_{\alpha_i}\}_{i=1}^N$ ($\alpha_i \in A$ for all i) such that $E \subset \bigcup_{i=1}^N U_{\alpha_i}$.

Remark. For any $E \subset X$, E has the subspace topology. So it should be a little bit careful when talking about open cover of E. If considering from its topology, an open cover of E is a collection $\{U_\alpha\}_{\alpha\in A}$ of subsets of E such that all U_α are *relatively open* in E and $E = \bigcup_{\alpha\in A} U_\alpha$. In the above definition, E is seen as a subset of X.

Question. Show that $\{(0, \frac{n}{n+1})\}_{n=1}^\infty$ is an open cover for $(0, 1)$ (as a subset of $\mathbb{R}$). Also, prove that it has no finite sub-cover.

We introduce the topological concept of compactness.

Definition 2.10 (Compact spaces). Let $K \subset X$.

(K1) X is said to be **compact** if for any open cover of X one can find a finite sub-cover, i.e. if $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X, then

$$\exists\ \{U_{\alpha_i}\}_{i=1}^N \ (\alpha_j \in A \text{ for all } j) \text{ such that } X = \textstyle\bigcup_{j=1}^N U_{\alpha_j}.$$

(K2) K is said to be **compact** if for any open cover of K one can find a finite subcover, i.e. if $\{U_\alpha\}_{\alpha\in A}$ is an open cover of K, then

$$\exists\ \{U_{\alpha_1}, \dots, U_{\alpha_N}\} \ (\alpha_j \in A \text{ for all } j) \text{ such that } K \subset \textstyle\bigcup_{j=1}^N U_{\alpha_j}.$$

Lemma 2.9. *Let $K \subset X$, where X is a metric space. We have K is compact as a subset of X if and only if K is compact as a metric space with the subspace topology.*

Proof. ($\Rightarrow$) Suppose K is compact as a subset of X. We have to show that K is compact as a metric space with the subspace topology from X.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of K by relatively open sets $U_\alpha \subset K$. By Lemma 2.8, $U_\alpha = U'_\alpha \cap K$ for some open set U'_α in X, for all $\alpha \in A$. Since $\{U_\alpha\}_{\alpha\in A}$ covers K, we have $K = \bigcup_{\alpha\in A} U_{\alpha\in A}$, it follows that $K \subset \bigcup_{\alpha\in A} U'_\alpha$. So $\{U'_\alpha\}_{\alpha\in A}$ is an open cover of K as a subset of X.

Now K is compact as a subset of X, one can find $\{U'_{\alpha_i}\}_{i=1}^N$ ($\alpha_i \in A$ for all i) such that $K \subset \bigcup_{i=1}^N U'_{\alpha_i}$. Thus $K \subset (\bigcup_{i=1}^N U'_{\alpha_i}) \cap K$, which implies $K = \bigcup_{i=1}^N U_{\alpha_i}$. We now conclude that K is compact as a metric space.

($\Leftarrow$) Suppose K is compact as a metric subspace of X. We have to show that K is compact as a subset of X.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of K, where all U_α are open sets in X. Let $U'_\alpha = U_\alpha \cap K$. Then $\{U'_\alpha\}_{\alpha\in A}$ is an open cover of the metric space K by relatively open sets $U'_\alpha \subset K$. Since K is compact as a metric space, $\exists$ a finite sub-cover $\{U'_{\alpha_i}\}_{i=1}^N$ for K. So $\{U_{\alpha_i}\}_{i=1}^N$ covers K. We conclude that K is compact as a subset of X. $\square$

Example 2.20. $\emptyset$ is compact.

If X a metric space that has finitely many elements, then X is compact.

$\mathbb{N}$ is non-compact (either as a subset of $\mathbb{R}$, or as a metric subspace of $\mathbb{R}$). ■

Question. Show that if X is an infinite set with the discrete metric, then it is non-compact.

Definition 2.11. A subset E of a metric space X is said to be **bounded** if there is an open ball $B_r(x_0)$ such that $E \subset B_r(x_0)$.

Example 2.21. A subset $E \subset \mathbb{R}$ is bounded provided there is $r > 0$ such that

$$\forall\, x \in E, |x| < r,$$

that is $E \subset (-r, r)$ for some $r > 0$.

Generally, a subset $E \subset \mathbb{R}^n$ is bounded provided there is $r > 0$ such that

$$\forall\, x \in E, \|x\| < r,$$

where $\|\cdot\|$ is the Euclidean norm. ■

Question. $\mathbb{R}, \mathbb{R}^n$ are unbounded metric spaces. However, they are bounded when equipped with the discrete metrics.

Theorem 2.8 (**A characterization of compact subsets**). *If K is a compact subset of a metric space then K is closed and bounded.*

In particular, if X is a compact metric space then there exists $r > 0$ and $x_0 \in X$ such that $d(x, x_0) < r$ for all $x \in X$.

Proof. Suppose K is compact.

(**Bounded**) We show that K is bounded. Fix $x_0 \in X$. Consider the collection of balls

$$\mathcal{B} = \{B_r(x_0) : r > 0\}.$$

Clearly, $\mathcal{B}$ covers K. Since K is compact, $\exists\, r_1, \ldots, r_N > 0$ such that $K \subset \bigcup_{j=1}^N B_{r_j}(x_0)$. In particular, let $r = \max\{r_j : 1 \leqslant j \leqslant N\}$, then $K \subset B_r(x_0)$. So K is bounded.

(**Closed**) Next we show that K is closed. Let $U = K^c$ and $x_0 \in U$. We have to show that there is $r > 0$ such that $B_r(x_0) \subset U$.

For each $x \in K$, we have

$$V_x \cap U_x = \emptyset^1 \quad \text{where } V_x := B_\delta(x_0), U_x := B_\delta(x) \text{ and } \delta = \tfrac{1}{2} d(x, x_0).$$

[1] If there were $z \in V_x \cap U_x$, then $d(x, x_0) \leqslant d(x, z) + d(z, x_0) < 2r = d(x, x_0)$ which is impossible.

Now $\{U_x : x \in K\}$ is an open cover of K, so $\exists\, U_{x_1}, \dots, U_{x_N}$ such that $K \subset \bigcup_{j=1}^N U_{x_j}$. Let $V = \bigcap_{j=1}^N V_{x_j}$. Then V is an open set and furthermore

$$V \cap K \subset \bigcup_{j=1}^{N} V \cap U_{x_j} \subset \bigcup_{j=1}^{N} V_{x_j} \cap U_{x_j} = \emptyset.$$

Thus $x_0 \in V \subset U$. Therefore U is an open set. □

Example 2.22. $\mathbb{R}$ and $\mathbb{Q}$ are unbounded metric spaces, so they are non-compact. ■

Question. Show that any intervals of the form $(a, b), (a, b], [a, b)$ are non-compact and so are $(a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$, where $a < b$.

Hint. Use that $\mathbb{R}$ is connected.

On an arbitrary metric space X, the converse of Theorem 2.8 may not be true.

Example 2.23. On the metric space $\mathbb{Q}$, the set

$$E = \{p \in \mathbb{Q} : p > 0, 2 < p^2 < 3\}$$

is closed and bounded. However, E is not compact (see Exercise 2.4). ■

Definition 2.12. A metric space X is said to have the **Heine-Borel property** if every closed and bounded subset of X is compact.

Theorem 2.9 (**Closed sets in a compact metric space**). *Assume X is a compact metric space. If $E \subset X$ is a closed set then E is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of E and $U = E^c$. Since E is a closed set, U is an open set. The collection $\{U\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover of X. Since X is compact, $\exists\, U_{\alpha_1}, \dots, U_{\alpha_N}$ such that

$$X = U \cup U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

Since $E \cap U = \emptyset$, we get that $E \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$. Thus E is compact. □

Proposition 4. *For any $a \leqslant b$, the closed interval $[a, b]$ is compact in $\mathbb{R}$.*

Proof. We prove by contradiction. Assume there were an open cover $\mathcal{U} := \{U_\alpha\}_{\alpha \in A}$ of $[a, b]$ such that any finite sub-collection does not cover $[a, b]$. We construct closed intervals $I_1, I_2, \dots$ as follows.

Let $c = (a + b)/2$. Since $\mathcal{U}$ has no finite sub-collection covering $[a, b]$, there is one among $[a, c], [c, b]$ that cannot be covered by any finite sub-collection of $\mathcal{U}$, call it I_1. Considering I_1 and applying the same argument, we get $I_2 \subset I_1$ that cannot be covered by any finite sub-collection of $\mathcal{U}$. Continuing the process, we obtain nested closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

such that each I_k cannot be covered by any finite sub-collection of $\mathcal{U}$. Note that length $I_k = (b-a)/2^k$, so

$$|x-y| \leqslant (b-a)/2^k \quad \forall\, x, y \in I_k.$$

Now we obtain by the nested interval theorem that there is $x_0 \in [a,b]$ such that

$$x_0 \in I_k \quad \text{for all } k.$$

Since $\mathcal{U}$ covers $[a,b]$, one can find U_α such that $x_0 \in U_\alpha$. Since U_α is an open set, there is $r > 0$ such that $(x_0 - r, x_0 + r) \subset U_\alpha$. Choose k so that $(b-a)/2^k < r$, then we get $I_k \subset (x_0 - r, x_0 + r) \subset U_\alpha$. So such I_k can be covered by the single set U_α, a contradiction. $\square$

Lemma 2.10 (Products of compact metric spaces). *Let X, Y be compact metric spaces. Then the product metric space $X \times Y$ is compact. Generally, if $X_1, \ldots, X_n$ are compact metric spaces, then so is $X_1 \times \cdots \times X_n$.*

Proof. We note that the second assertion follows from the first one.

Let $\mathcal{W}$ be an open cover of $X \times Y$. We have to show that $\mathcal{W}$ contains a finite sub-cover of $X \times Y$.

Let $x_0 \in X$. Let us show that there is an open set U_{x_0} containing x_0 and a finite sub-collection $\mathcal{W}_{x_0} \subset \mathcal{W}$ such that

$$U_{x_0} \times Y \subset \bigcup \mathcal{W}_{x_0}.$$

To see this, let $y \in Y$ and take $W \in \mathcal{W}$ such that $(x_0, y) \in W$. Since W is open in $X \times Y$, $\exists\, r = r(y) > 0$ such that $B_{2r}^{X\times Y}(x_0, y) \subset W$. By the triangle inequality, then

$$B_r^X(x_0) \times B_r^Y(y) \subset B_{2r}^{X\times Y}(x_0, y) \subset W.$$

The collection $\{B_{r(y)}^Y(y) : y \in Y\}$ is an open cover of Y. Since Y is compact, there exist $y_1, \ldots, y_N$ and corresponding $r_1, \ldots, r_N$, $W_1, \ldots, W_N \in \mathcal{W}$ such that $\bigcup_{j=1}^N B_{r_j}^Y(y_j) = Y$ and also

$$B_{r_j}^X(x_0) \times B_{r_j}^Y(y_j) \subset W_j, \quad j = 1, \ldots, N.$$

Taking $U_{x_0} = B_r^X(x_0)$ where $r = \min\{r_1, \ldots, r_N\}$ and $\mathcal{W}_{x_0} = \{W_1, \ldots, W_N\}$ we get $U_{x_0} \times Y \subset \bigcup_{j=1}^N (B_{r_j}^X(x_0) \times B_{r_j}^Y(y_j)) \subset \bigcup_{j=1}^N W_j = \bigcup \mathcal{W}_{x_i}$.

Next we consider the collection $\{U_x : x \in X\}$. This is an open cover of X. Since X is compact, there are $\{U_{x_1}, \ldots, U_{x_n}\}$ that covers X. Take the corresponding $\mathcal{W}_{x_1}, \ldots, \mathcal{W}_{x_n}$ of finite sub-collection of $\mathcal{W}$ such that

$$U_{x_j} \times Y \subset \bigcup \mathcal{W}_{x_j} \quad j = 1, \ldots, n.$$

Taking $\mathcal{W}' = \{W : W \in \mathcal{W}_{x_j} \text{ for some } j = 1, \ldots, K\}$, we obtain a finite sub-collection $\mathcal{W}'$ of $\mathcal{W}$ such that

$$X \times Y = \bigcup_{j=1}^{K} U_{x_j} \times Y \subset \bigcup \mathcal{W}'.$$

So $\mathcal{W}'$ is a finite sub-cover of $X \times Y$. Hence $X \times Y$ is compact. □

Remark. The converse of Lemma 2.10 is also true. That is if $X \times Y$ is compact, then X and Y are compact. See Exercise 2.4.

Theorem 2.10 (Heine-Borel). *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. In other word, $\mathbb{R}^n$ has the Heine-Borel property.*

Proof. ($\Rightarrow$) This is Theorem 2.8.

($\Leftarrow$) Let E be a closed and bounded subset of $\mathbb{R}^n$. It suffices to show that every cube

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

is compact, where $a_1 \leqslant b_1, \ldots, a_n \leqslant b_n$ are real numbers. In fact, since E is bounded, it contains in some cube Q. Now if we can show that every cube is compact, then using that E is closed, we get by Theorem 2.9 that E is compact.

Now every cube is compact by the preceding proposition and lemma. □

Exercise 2.4.

1. *In $\mathbb{R}$, show that $\{\frac{1}{n} : n \in \mathbb{N}\}$ is non-compact.*
2. *Show that in $\mathbb{R}^n$, every straight line is non-compact.*
3. *Prove that $E = \{p \in \mathbb{Q} : 2 \leqslant p^2 \leqslant 3, p > 0\}$ is not compact.*
4. *Let E be a closed subset of X and K is a compact subset. Prove that $E \cap K$ is compact.*
5. *Prove that if X, Y are metric spaces and $X \times Y$ is compact, then so are X, Y.*
6. *Prove that a finite union of compact metric spaces is compact.*
7. *If X is a metric space having the property that every closed ball is compact, then X is a space with the Heine-Borel property.*

■

Mathematical Analysis: Lecture 7

Sujin Khomrutai, Ph.D.

Summary. *In this lecture we study*

1. *Sequences in a metric space*
2. *Convergence/divergence and limits*
3. *Some properties of convergent sequences*

3 Sequences and Series

In this chapter we study the concepts of sequences and series. In the discussion of sequences, it is not too difficult to consider sequences in a metric space. Most of our examples, are sequences and series in $\mathbb{R}$ or $\mathbb{R}^n$.

3.1 Convergent (or divergent) sequences

Throughout this chapter, (X, d) is a metric space. $\mathbb{R}$ and $\mathbb{R}^n$ are assumed to have the standard and Euclidean metrics, respectively.

By a **sequence in** X we mean an arrangement of elements of X:

$$\{x_1, x_2, \ldots, x_n, \ldots\}.$$

It will be denoted by

$$\{x_n\}_{n=1}^{\infty} \text{ or } \{x_n\} \text{ for simplicity.}$$

Example 3.1. We present some quick examples.

1. Let $x_0 \in X$. The sequence
$$\{x_0, x_0, x_0, \ldots\}$$
is called a *constant sequence*.

2. In $\mathbb{R}$, there are some well-known sequences such as harmonic sequence:

$$\{1/n\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

geometric sequence:

$$\{ar^n\}_{n=1}^{\infty} = \{a, ar, ar^2, ar^3, \dots\},$$

arithmetic sequence:

$$\{a + (n-1)d\}_{n=1}^{\infty} = \{a, a+d, a+2d, a+3d, \dots\},$$

a sequence defining e:

$$\{(1+1/n)^n\}_{n=1}^{\infty} = \left\{2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \dots\right\}$$

3. $\{\frac{n}{n+2}\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$, and $\{(n, \sin n)\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^2$.

4. One can also consider sequences that have starting index other than 1, e.g.

$$\{2 \cdot 3^n\}_{n=0}^{\infty}, \quad \{x_n\}_{n=N}^{\infty} = \{x_N, x_{N+1}, x_{N+2}, \dots\}$$

where $N \in \mathbb{N}$.

5. Sometimes, we may need to use a difference index such as $\{x_k\}_{k=1}^{\infty}$.

■

Some sequences are defined inductively.

Example 3.2 (Fibonacci sequence)**.** Let $F_0 = F_1 = 1$. Define for each $n > 1$,

$$F_n = F_{n-1} + F_{n-2}.$$

Then $\{F_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$. Without considering the initial values F_0, F_1, one can formally find solution of the form $F_n = Cr^n$. Then

$$Cr^n = Cr^{n-1} + Cr^{n-2} \quad \Rightarrow \quad r^2 = r + 1.$$

That gives two numbers $r_1 = (1+\sqrt{5})/2, r_2 = (1-\sqrt{5})/2$. Thus generally, we obtain

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Setting $F_0 = F_1 = 1$, $A = 1/\sqrt{5}, B = -1/\sqrt{5}$. ■

Definition 3.1. Let $\{x_n\}$ be a sequence in X. A **subsequence** of $\{x_n\}$ is a sequence of the form

$$\{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\} = \{x_{n_k}\}_{k=1}^{\infty}$$

where $n_1 < n_2 < \cdots < n_k < \cdots$ are positive integers.

Example 3.3. For a sequence $\{x_n\}$, we list some subsequences

$$\begin{aligned} &\{x_2, x_4, x_6, \dots, x_{2k}, \dots\} = \{x_{2k}\}_{k=1}^{\infty} \\ &\{x_3, x_9, \dots, x_{3^k}, \dots\} = \{x_{3^k}\}_{k=1}^{\infty} \\ &\{x_N, x_{N+1}, \dots\} \end{aligned}$$

where $N \in \mathbb{N}$. ■

Definition 3.2. Let $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ is said to be **convergent in** X if $\exists\, a \in X$ such that

   $$\forall \varepsilon > 0, \exists\, N = N_\varepsilon \in \mathbb{N} \text{ such that if } n \geqslant N \text{ then } d(a, x_n) < \varepsilon.$$

   a (which is unique, see below) is called the **limit** of the sequence. If $\{x_n\}$ is convergent and a is the limit, we also say that it **converges to** a, written

   $$x_n \to a \quad \text{in } X, \text{ or } \lim_{n\to\infty} x_n = a$$

2. If there is no element $a \in X$ such that the above statement is true, we say that $\{x_n\}$ is **divergent** or it **diverges**.

Lemma 3.1 (Uniqueness of limit). *If $\{x_n\}$ is convergent then it converges to a unique element in X.*

Proof. Suppose $a, b \in X$ are such that $x_n \to a$ and $x_n \to b$ in X.

($\varepsilon/2$ trick!) Let $\varepsilon > 0$. Since $x_n \to a$, there is a positive integer N such that

$$n \geqslant N \quad \Rightarrow \quad d(x_n, a) < \varepsilon/2.$$

Similarly, since $x_n \to b$, there is N' such that

$$n \geqslant N' \quad \Rightarrow \quad d(x_n, b) < \varepsilon/2.$$

Choose $n \in \mathbb{N}$ such that $n \geqslant \max\{N, N'\}$. By the triangle inequality we have

$$d(a, b) \leqslant d(a, x_n) + d(x_n, b) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So we get $d(a, b) < \varepsilon$ for all $\varepsilon > 0$, hence $d(a, b) = 0$, that is $a = b$. □

Remark. In $\mathbb{R}$, that $x_n \to a$ can be expressed as

$$\forall \varepsilon > 0, \exists N = N_\varepsilon \in \mathbb{N} \text{ such that if } n \geqslant N \text{ then } |x_n - a| < \varepsilon. \tag{$\mathbb{R}$}$$

In $\mathbb{R}^m$ $(m \geqslant 2)$, that $\vec{x}_n \to \vec{a}$ can be expressed as

$$\forall \varepsilon > 0, \exists N = N_\varepsilon \in \mathbb{N} \text{ such that if } n \geqslant N \text{ then } \|\vec{x}_n - \vec{a}\| < \varepsilon. \tag{$\mathbb{R}^m$}$$

Here $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$.

Question. Show that every constant sequence converges. Furthermore, every *eventually constant sequence*, i.e. $\{x_n\}$ such that $\exists\ N \in \mathbb{N}$, $x_n =$ a constant, for all $n \geqslant N$, converges.

Example 3.4. Prove that $\dfrac{1}{n} \to 0$ in $\mathbb{R}$.

Solution. Let $\varepsilon > 0$. Since $1/\varepsilon \in \mathbb{R}$, we can choose $N \in \mathbb{N}$ such that $1/\varepsilon \leqslant N$, i.e. $1/N \leqslant \varepsilon$. If $n \geqslant N$, then we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leqslant \frac{1}{N_\varepsilon} \leqslant \varepsilon.$$

Thus $1/n \to 0$. ∎

Example 3.5. Prove that $\displaystyle\lim_{n\to\infty} \frac{2n-1}{n+7} = 2$ in $\mathbb{R}$.

Solution. Let $\varepsilon > 0$. Note that $|\frac{2n-1}{n+7} - 2| < \varepsilon \Leftrightarrow \frac{15}{n+7} < \varepsilon$. So take $N \in \mathbb{N}$ such that $N \geqslant \frac{15}{\varepsilon} - 7$. Then for $n \geqslant N$, we get that $|\frac{2n-1}{n+7} - 2| < \varepsilon$. ∎

Lemma 3.2 (**Criterion for divergence**)**.** *A sequence $\{x_n\}$ in a metric space X is divergent if and only if*

$$\forall\, a \in X, \exists\, \varepsilon > 0 \text{ and } \{x_{n_k}\}_{k=1}^\infty \text{ such that } d(a, x_{n_k}) \geqslant \varepsilon \text{ for all } k.$$

Proof. Obvious. □

Example 3.6. Show $\{(-1)^n\}_{n=1}^\infty$ is divergent in $\mathbb{R}$.

Solution. Let $a \in \mathbb{R}$.

If $a \neq -1$, we choose any $0 < \varepsilon \leqslant |1 + a|$. Then $|(-1)^{2k+1} - a| \geqslant \varepsilon$ for all $k \in \mathbb{N}$.

If $a \neq 1$, we choose any $0 < \varepsilon \leqslant |1 - a|$. Then $|(-1)^{2k} - a| \geqslant \varepsilon$ for all $k \in \mathbb{N}$. We conclude by the above lemma that the sequence diverges. ∎

Example 3.7. Let (X, δ) be a non-empty set with the discrete metric. Show that $\{x_n\}$ is convergent in X if and only if it is eventually constant.

Solution. $(\Rightarrow)$ Suppose $x_n \to a$ in X. For $\varepsilon = 1/2$, $\exists\, N \in \mathbb{N}$ such that $\delta(x_n, a) < 1/2$, so $d(x_n, a) = 0$. This means $x_n = a$ for all $n \geqslant N$.

$(\Leftarrow)$ See the preceding **Question**. ∎

Lemma 3.3 (Every convergent sequence is bounded). *If $\{x_n\}$ converges in X, then it is bounded. So, if a sequence is unbounded, then it diverges.*

Proof. Suppose $x_n \to a$ in X. We have to show that the whole sequence is contained in a ball.

For $\varepsilon = 1$, since $x_n \to a$, $\exists\, N \in \mathbb{N}$ such that if $n \geqslant N$ then $d(a, x_n) < 1$. Choose $r > \max\{d(a, x_1), \ldots, d(a, x_{N-1}), 1\}$. Then $d(x_n, a) < r$ for all $n \in \mathbb{N}$, which means $\{x_n\}$ is bounded. □

Example 3.8. Show that $\{x_n = n\}_{n=1}^{\infty}$ is divergent.

Solution. Since $\{x_n = n\}_{n=1}^{\infty}$ is unbounded, it diverges. ■

In proving convergences, the requirement $d(a, x_n) < \varepsilon$ can be relaxed a bit according to the following lemma.

Lemma 3.4 (Verify $d(a, x_n) < L\varepsilon$ instead of $d(a, x_n) < \varepsilon$)**.** *Let $L > 0$ be a constant. We get $x_n \to a$ if and only if*

$$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \text{ such that if } n \geqslant N, \text{ then } d(a, x_n) < L\varepsilon.$$

Proof. See Exercise □

Theorem 3.1 (Algebraic properties of limit). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Suppose $x_n \to a$ and $y_n \to b$. Then*

1. *(Sum or difference) $x_n + y_n \to a + b$ and $x_n - y_n \to a - b$.*
2. *(Scalar multiple) If c is a constant then $cx_n \to ca$.*
3. *(Product) $x_n y_n \to ab$*
4. *(Quotient) If $b \neq 0$ then $\dfrac{x_n}{y_n} \to \dfrac{a}{b}$.*

Proof. 1. Let $\varepsilon > 0$. Since $x_n \to a$, $\exists\, N \in \mathbb{N}$ such that if $n \geqslant N$ then $|x_n - a| < \varepsilon$. Similarly, since $y_n \to b$, $\exists\, N'$ such that if $n \geqslant N'$ then $|x_n - b| < \varepsilon$. By the triangle inequality, if $n \geqslant \max\{N, N'\}$ then

$$|(x_b + y_n) - (a + b)| = |(x_n - a) + (y_n - b)| \leqslant |x_n - a| + |y_n - b| < 2\varepsilon.$$

So $x_n + y_n \to a + b$ by Lemma 3.4 (with $L = 2$).

2. If $c = 0$, we have a constant sequence $0 \cdot x_n \to 0 \cdot a$. Assume $c \neq 0$. Let $\varepsilon > 0$. Since $x_n \to a$, $\exists\, N \in \mathbb{N}$ such that if $n \geqslant N$ then $|x_n - a| < \varepsilon$. This implies

$$|(cx_n) - (ca)| = |c||x_n - a| < |c|\varepsilon.$$

So $cx_n \to ca$ by Lemma 3.4 (with $L = |c|$).

3. We shall use the identity $x_n y_n = (x_n - a)(y_n - b) + a(y_n - b) + b(x_n - a) + ab$. Define sequences $u_n = (x_n - a)(y_n - b), v_n = a(y_n - b)$, and $w_n = b(x_n - a)$. So

$$x_n y_n = u_n + v_n + w_n + ab.$$

It is directly to verify, using Lemma 3.4, that $v_n \to 0, w_n \to 0$. Also, $ab \to ab$ for a constant sequence. So, in view of the sum law, it suffices to show that $u_n \to 0$.

Let $\varepsilon > 0$. Since $x_n \to a$, $\exists\ N \in \mathbb{N}$ such that if $n \geqslant N$ then $|x_n - a| < \sqrt{\varepsilon}$. Since $y_n \to b$, $\exists\ N'$ such that if $n \geqslant N'$ then $|y_n - b| < \sqrt{\varepsilon}$. Now for $n \geqslant \max\{N, N'\}$,

$$|(x_n - a)(y_n - b) - 0| = |x_n - a||y_n - b| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

Thus $u_n \to 0$ as claim. So $x_n y_n \to ab$.

4. By 3, it suffices to prove that $\dfrac{1}{y_n} \to \dfrac{1}{b}$. We use the identity

$$\left|\frac{1}{y_n} - \frac{1}{b}\right| = \frac{|y_n - b|}{|y_n||b|}$$

Since $y_n \to b$ and $b \neq 0$, $\exists\ N \in \mathbb{N}$ such that if $n \geqslant N$ then $|y_n - b| \leqslant \frac{1}{2}|b|$. So

$$|y_n| \geqslant |b| - |y_n - b| \geqslant \frac{1}{2}|b|.$$

Let $\varepsilon > 0$. Since $y_n \to b$, $\exists\ N' \in \mathbb{N}$ such that if $n \geqslant N'$ then

$$|y_n - b| < \varepsilon.$$

If $n \geqslant \max\{N, N'\}$, we get

$$\left|\frac{1}{y_n} - \frac{1}{b}\right| = \frac{|y_n - b|}{|y_n||b|} \leqslant \frac{2|y_n - b|}{|b|^2} < \frac{2}{|b|^2}\varepsilon.$$

We conclude that $\dfrac{1}{y_n} \to \dfrac{1}{b}$by Lemma 3.4 (with $L = 2/|b|^2$). □

Using the preceding theorem and that $1/n \to 0$, we can find the limits of many sequences in $\mathbb{R}$.

Example 3.9. Determine whether each of the following sequences is convergent/divergent If it converges, find the limit.

1. $\left\{x_n = \dfrac{2n+3}{3n+5}\right\}$

2. $\left\{y_n = \dfrac{4n}{5n^2+1}\right\}$

Solution. (1) We express

$$x_n = \frac{2 + 3/n}{3 + 5/n}.$$

Since $2 \to 2, 3/n \to 0, 3 \to 3$, and $5/n \to 0$, we get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2 + 3/n}{3 + 5/n} = \frac{2}{3}.$$

(2) We express

$$y_n = \frac{4/n}{5 + 1/n^2}.$$

Since $4/n \to 0, 5 \to 5$, and $1/n^2 = (1/n)(1/n) \to 0 \cdot 0$, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{4/n}{5 + 1/n^2} = \frac{0}{5+0} = 0.$$

■

Lemma 3.5 (Subsequence criterion for convergence). *In a metric space X, a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a if and only if every subsequence converges to a.*

In particular, if there is a subsequence that does not converge or there are two subsequences that converge to different elements, then $\{x_n\}$ diverges.

Proof. ($\Rightarrow$) Assume $x_n \to a$. Let $\{x_{n_k}\}$ be a subsequence. Let $\varepsilon > 0$. Since $x_n \to a$, $\exists\ N_\varepsilon \in \mathbb{N}$ such that if $n \geqslant N_\varepsilon$ then

$$d(x_n, a) \leqslant \varepsilon.$$

Let $K_\varepsilon \in \mathbb{N}$ be such that if $k \geqslant K$ then $n_k \geqslant N_\varepsilon$. If $k \geqslant K_\varepsilon$ then

$$d(x_{n_k}, a) \leqslant \varepsilon.$$

Thus $x_{n_k} \to a$.

($\Leftarrow$) The statement is obvious since $\{x_n\}_{n=1}^{\infty}$ in itself is a subsequence. □

Example 3.10. Prove that $\{x_n = (-1)^n\}_{n=1}^{\infty}$ diverges.

Solution. Consider two subsequences

$$x_{2k} = (-1)^{2k} = 1, \quad x_{2l+1} = (-1)^{2l+1}$$

for $k = 1, 2, \ldots$ and $l = 0, 1, \ldots$. It follows that $x_{2k} \to 1$ and $x_{2l+1} \to -1$, so $\{x_n\}$ diverges.

■

Exercise 3.1.

1. *Prove the following generalization of Lemma 3.4. Suppose* L, M *are positive constants. We have* $x_n \to a$ *in* X *if and only if*

$$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \text{ such that if } n \geqslant N, \text{ then } d(a, x_n) < L\varepsilon^M.$$

2. *Suppose* $x_n \to a$ *in* $\mathbb{R}$. *Prove the following results.*

   (a) $x_n^k \to a^k$ $(k \in \mathbb{N})$

   (b) *If* $a > 0$ *and* $x_n > 0$ *for all* n, *then* $x_n^{1/k} \to a^{1/k}$. $(k \in \mathbb{N})$. *In particular,* $\sqrt{x_n} \to \sqrt{a}$.

   Hint. You may need Lemma 3.3 and 3.4.

3. *Use the* ε-N *definition (or Lemma 3.4) to prove the following results.*

   (a) *For* $-1 < r < 1$, $r^n \to 0$ *in* $\mathbb{R}$.

   (b) *Suppose* $\{x_n\}, \{y_n\}$ *are sequences in* $\mathbb{R}$, $\{x_n\}$ *is bounded and* $y_n \to 0$. *Then* $x_n y_n \to 0$. *(Example:* $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.*)*

   (c) *Suppose* $x_n \to a$ *in* $\mathbb{R}$. *Then*

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = a.$$

4. *Prove that if* $\lim\limits_{n\to\infty} (x_{n+1} - x_n) = a$ *in* $\mathbb{R}$, *then* $\lim\limits_{n\to\infty} \frac{x_n}{n} = a$ *in* $\mathbb{R}$.

   Hint. Use 3 (c) above and telescoping.

Mathematical Analysis: Lecture 8

Sujin Khomrutai, Ph.D.

Summary. *We study*

1. *The squeeze theorem and monotone sequences*
2. *Cauchy sequences and Cauchy criterion.*
3. *Bolzano-Weierstrass theorem.*
4. *Cauchy completeness of $\mathbb{R}$ (∗) and $\mathbb{R}^n$.*
5. *(∗) Every compact metric space is complete.*

3.2 Monotone sequences and the squeeze theorem

To determine whether a given sequence in $\mathbb{R}$ is convergent or divergent, it is often useful to compare the given sequence with some other sequences which are better behave.

Theorem 3.2 (The squeeze theorem). *Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences in $\mathbb{R}$. If $y_n \leqslant x_n \leqslant z_n$ for all n and $y_n \to a$, $z_n \to a$, then $x_n \to a$.*

Proof. Let $\varepsilon > 0$. Since $y_n \to a$, $\exists\ N \in \mathbb{N}$ such that if $n \geqslant N$ then $|y_n - a| < \varepsilon$. Similarly, since $z_n \to a$, $\exists\ N' \in \mathbb{N}$ such that if $n \geqslant N'$ then $|z_n - a| < \varepsilon$. For $n \geqslant \max\{N, N'\}$ we have

$$|x_n - a| \leqslant \max\{|y_n - a|, |z_n - a|\} < \varepsilon.$$

So $x_n \to a$. □

Example 3.11. Show that the following sequences converge.

1. $x_n = \dfrac{(-1)^n}{n}$

2. $y_n = \dfrac{n + \sin n}{2n+1}$

Solution. We have

$$-\frac{1}{n} \leqslant x_n \leqslant \frac{1}{n}$$

and since $1/n \to 0, -1/n \to 0$, we conclude that $x_n \to 0$ by the squeeze theorem.

(2) We have

$$\frac{n-1}{2n+1} \leqslant y_n \leqslant \frac{n+1}{2n+1}$$

and since $(n-1)/(2n+1) \to 1/2$, $(n+1)/(2n+1) \to 1/2$, we get $y_n \to 1/2$.

Alternatively, one may observe that $y_n = (1+(\sin n)/n)/(2+1/n)$. Since $|(\sin n)/n| \leqslant 1/n$, it follows that $(\sin n)/n \to 0$. So $y_n \to (1+0)/(2+0) = 1/2$. ∎

Question. Employing the squeeze theorem, show that, in $\mathbb{R}$, $x_n \to 0$ if and only if $|x_n| \to 0$. In fact, generally, $x_n \to a \Leftrightarrow |x_n - a| \to 0$.

Example 3.12. Let $a > 0$. Prove that

$$\lim_{n\to\infty} a^{1/n} = 1.$$

Solution. The case $a = 1$ is trivial. If $0 < a < 1$, we show instead that $(1/a)^{1/n} \to 1$. So now we can assume that $a > 1$.

Let $x_n = a^{1/n} - 1$. We have to show that $x_n \to 0$. Since $a > 1$, it follows that $x_n \geqslant 0$ for all n. We rewrite and employ the binomial theorem to get that

$$a = (1+x_n)^n = 1 + \binom{n}{1} x_n + \binom{n}{2} x_n^2 + \cdots + \binom{n}{n} x_n^n \geqslant \binom{n}{2} x_n^2.$$

The obtained inequality implies that $x_n \leqslant [\frac{2a}{n(n-1)}]^{1/2}$. Since $[\frac{2a}{n(n-1)}]^{1/2} \to 0$, we apply the squeeze theorem to conclude that $x_n \to 0$. Thus $a^{1/n} \to 1$. ∎

Question. Suppose $x_n \to a, y_n \to b$ in $\mathbb{R}$. Use the squeeze theorem to show that

$$(|x_n|^n + |y_n|^n)^{1/n} \to \max\{|a|, |b|\}.$$

Definition 3.3 (Monotone sequences). Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

1. $\{x_n\}$ is called **increasing** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$.
2. $\{x_n\}$ is called **strictly increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.
3. $\{x_n\}$ is called **decreasing** if $x_{n+1} \leqslant x_n$ for all $n \in \mathbb{N}$.
4. $\{x_n\}$ is called **strictly decreasing** if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

A sequence $\{x_n\}$ which is either increasing or decreasing is said to be **monotone**.

We have seen in the previous lecture that every convergent sequence is bounded, the converse is not true, e.g. $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it does not converge. For monotonic sequences, being convergence or bounded give the same thing.

Theorem 3.3 (**Monotone sequence theorem**). *Let $\{x_n\}$ be a monotone sequence in $\mathbb{R}$. Then $\{x_n\}$ is convergent if and only if it is bounded.*

Proof. ($\Rightarrow$) If $\{x_n\}$ converges then it is bounded by Lemma 3.3.

($\Leftarrow$) It suffices to prove the result when $\{x_n\}$ is increasing. In fact, for the other case we just consider $\{-x_n\}$.

Suppose $\{x_n\}$ is a bounded increasing sequence in $\mathbb{R}$. By the completeness property of $\mathbb{R}$ (Theorem 1.2), since $\{x_n\}$ is bounded, we get a real number

$$a = \sup E, \quad E = \{x_n : n \in \mathbb{N}\} \text{ (as a set)}.$$

We will show that $x_n \to a$.

Clearly, $x_n \leqslant a$ for all n. Let $\varepsilon > 0$. Since $a = \sup E$, there is N such that $a - \varepsilon < x_N \leqslant a$. For $n \geqslant N$, since $\{x_n\}$ is increasing, we get

$$a - \varepsilon < x_N \leqslant x_n \leqslant a \quad \Rightarrow \quad |x_n - a| \leqslant \varepsilon.$$

This implies $x_n \to a$. $\square$

Remark.

1. If $\{x_n\}$ is a bounded increasing in $\mathbb{R}$, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.
2. If $\{x_n\}$ is a bounded decreasing in $\mathbb{R}$, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.

Example 3.13. Let $x_n \to a$ in $\mathbb{R}$. Define the sequence $y_n = \max\{x_1, \ldots, x_n\}$ for all $n \in \mathbb{N}$. Prove that $\{y_n\}$ converges and find the limit.

Solution. Clearly, $y_{n+1} \leqslant y_n$, so $\{y_n\}$ is increasing. We show that $\{y_n\}$ is bounded. Since $x_n \to a$, $\{x_n\}$ is bounded. So there is $M \in \mathbb{R}$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Thus $|y_n| \leqslant M$ as well. It follows by the monotone sequence theorem that $\{y_n\}$ is convergent. According to the preceding remark, the limit is $\sup\{x_n : n \in \mathbb{N}\}$. ■

Question. Let $x_n \to a$ in $\mathbb{R}$. Define the sequence

$$z_n = \min\{x_k : k = 1, 2, \ldots, n\} \quad (n \in \mathbb{N}).$$

Prove that $\{z_n\}$ converges and find the limit.

Example 3.14. Let $x_1 = 1$ and

$$x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n} \quad \text{for } n > 0.$$

Prove that the sequence is increasing and is bounded above by $\sqrt{2}$. Also find the limit.

Solution. We prove by induction the second statement that $x_n \leqslant \sqrt{2}$ for all $n > 0$. The basis step is obvious.

Suppose $x_n \leqslant \sqrt{2}$. Consider

$$x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n} = \frac{3}{2} - \frac{1/2}{3 + 2x_n} \leqslant \frac{3}{2} - \frac{1/2}{3 + 2\sqrt{2}} = \sqrt{2}.$$

Thus we get by induction that $x_n \leqslant \sqrt{2}$ for all n.

Let us prove that $\{x_n\}$ is increasing. Consider

$$x_{n+1} - x_n = \frac{4 + 3x_n}{3 + 2x_n} - x_n = \frac{4 - 2x_n^2}{3 + 2x_n} \geqslant 0,$$

because $x_n \leqslant \sqrt{2}$. Let $a = \lim_{n\to\infty} x_n$. Taking $n \to \infty$ into the given identity and applying the limit laws, we get

$$a = \frac{4 + 3a}{3 + 2a} \quad \Rightarrow \quad 4 + 3a = 3a + 2a^2 \quad \Rightarrow \quad a = \sqrt{2}.$$

where we have used that $a > 0$. ■

3.3 Cauchy sequences

Now we introduce a central concept in mathematical analysis, the *completeness in the sense of Cauchy*. This concept is defined for metric spaces, unlike the completeness property in the first chapter which only defined for ordered fields.

As a motivation to the following theorem, let us consider the following example. We have, for $x_n = \frac{n}{n+1}$ that $x_n \to 1$. By the triangle inequality,

$$|x_n - x_m| \leqslant |x_n - 1| + |x_m - 1|.$$

So by taking $N \in \mathbb{N}$ such that $|x_n - 1| < \varepsilon/2$ and $|x_m - 1| < \varepsilon/2$ for all $n, m \geqslant N$, we have $|x_n - x_m| < \varepsilon$.

Definition 3.4. Let (X, d) be a metric space. A sequence $\{x_n\}$ in a metric space X is called a **Cauchy sequence** if

$$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \text{ such that if } n > m \geqslant N \text{ then } d(x_n, x_m) < \varepsilon.$$

This is called the **Cauchy condition**.

Lemma 3.6 (**Every convergent sequence is Cauchy**). *If $\{x_n\}$ is convergent in X, then $\{x_n\}$ is a Cauchy sequence.*

Question. Prove the preceding lemma.

The converse to the preceding lemma is not true, that is there is a non-convergent sequence Cauchy sequence in a certain metric space. See Exercise 3.2.

Lemma 3.7 (**Every Cauchy sequence is bounded**). *If $\{x_n\}$ is a Cauchy sequence in a metric space X, then $\{x_n\}$ is bounded, i.e. $\exists\, r > 0$ and $x_0 \in X$ such that $d(x_0, x_n) < r$ for all $n \in \mathbb{N}$.*

Proof. See Exercise 3.2. □

Question. Show that $\{x_n = (-1)^n\}$ and $\{y_n\} = \{\frac{n^2}{n+1}\}$ are not Cauchy sequences.

Remark.

1. Observe that for a sequence to be convergent, one needs to know the limit in advance. The Cauchy condition, however, does not involve any elements other than the terms of the sequence. As will be revealed, for "complete" metric spaces, the converse of Lemma 3.6 is also true, i.e. every Cauchy sequence converges. So for such a space, verifying the Cauchy condition is easier and turn out to be a powerful tool.

2. In the condition for Definition 3.4, the inequality $d(x_n, x_m) < \varepsilon$ can be relaxed to require $d(x_n, x_m) < L\varepsilon$, where L is a constant.

Example 3.15. Show that $\{1/n^2\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution. Let $x_n = 1/n^2$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/N^2 < \varepsilon$. Then for $n > m \geqslant N$ we have

$$|x_n - x_m| = \frac{n^2 - m^2}{n^2 m^2} \leqslant \frac{2n^2}{n^2 m^2} \leqslant \frac{2}{N^2} < 2\varepsilon.$$

Thus $\{x_n = 1/n^2\}$ is a Cauchy sequence. ■

Lemma 3.8. *Let $r \in (0, 1)$ and $M > 0$. Suppose $\{x_n\}$ is a sequence in a metric space such that*

$$d(x_{n+1}, x_n) \leqslant M r^n \quad \text{for all } n \geqslant 1.$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. We know that the infinite sum $r^N + r^{N+1} + r^{N+2} + \cdots = r^N/(1 - r)$. For $n > m \geqslant N$, we have by the triangle inequality and telescoping

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leqslant M r^m + M r^{m+1} + \cdots + M r^{n-1} \\ &< M(r^N + r^{N+1} + r^{N+2} + \cdots) = \frac{M r^N}{1 - r}. \end{aligned}$$

For each $\varepsilon > 0$, we choose N such that $Mr^N/(1-r) < \varepsilon$, this can be achieved because $r^N \to 0$ as $N \to \infty$. Then for such N, we obtain by the above estimate that $d(x_n, x_m) < \varepsilon$, i.e. $\{x_n\}$ is a Cauchy sequence. □

Example 3.16. Let $a, b \in \mathbb{R}$. Define a sequence $\{x_n\}$ by $x_1 = a, x_2 = b$ and

$$x_{n+1} = \frac{x_n + x_{n-1}}{2} \quad \text{for } n \geqslant 2.$$

Prove that $\{x_n\}$ is a Cauchy sequence.

Solution. The case $a = b$ is trivial. So we assume $a \neq b$. Observe that $x_3 - x_2 = -(x_2 - x_1)/2 = -(b-a)/2$, $x_4 - x_3 = -(x_3 - x_2)/2 = (-1)^2(b-a)/2^2$, and hence by induction we get

$$x_n - x_{n-1} = (-1)^{n-2}(b-a)/2^{n-2} \quad (n \geqslant 2).$$

This gives

$$|x_{n+1} - x_n| = 4|b-a|(1/2)^n.$$

The result then follows by the preceding example ($K = 4|b-a|, r = 1/2$). ■

Definition 3.5. A metric space is said to be **complete** if every Cauchy sequence in the space converges.

Remark. It can be shown that every metric space X can be included in a complete metric space $\bar{X}$ "continuously" so that every Cauchy sequence in X converges in $\bar{X}$.

The following lemma is one of the most powerful way to show a metric space is complete.

Lemma 3.9 (Every Cauchy sequence with a convergent subsequence converges). *Let $\{x_n\}$ be a Cauchy sequence in a metric space X. If there is subsequence $\{x_{n_k}\}$ that converges to $a \in X$, then $\{x_n\}$ is convergent and $x_n \to a$.*

Proof. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, $\exists\, N \in \mathbb{N}$ such that if $n > m \geqslant N$ then $d(x_n, x_m) < \varepsilon$. Since $x_{n_k} \to a$, $\exists\, K \in \mathbb{N}$ such that if $k \geqslant K$ then $d(x_{n_k}, a) < \varepsilon$.

Consider $n \geqslant N$. Then we choose $m = n_k$ for some $k \geqslant K$ such that $n_k \geqslant N$, then it follows by the triangle inequality that

$$d(x_n, a) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, a) < 2\varepsilon.$$

Thus we conclude that $\{x_n\}$ is convergent and furthermore $x_n \to a$. □

Before proving the following result, we recall that a sequence $\{x_n\}$ is bounded in $\mathbb{R}$ provided there is $r > 0$ such that $|x_n| < r$ for all $n \in \mathbb{N}$.

Theorem 3.4 (**Bolzano-Weierstrass**)**.** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Choose an interval $[a, b]$ such that $x_n \in [a, b]$ for all $n \in \mathbb{N}$. We have to show that there is a subsequence $\{x_{n_k}\}$ that converges to some $\alpha \in [a, b]$.

We employ the argument in the proof of the Heine-Borel theorem (Theorem 2.10). There is an interval I_1 among the sub-intervals $[a, c], [c, b]$, where $c = (a + b)/2$, that contains infinitely many terms of the sequence $\{x_n\}$. Choose a term $x_{n_1} \in I_1$.

Bisecting I_1, then, by the same reasoning, there is I_2 among the two sub-intervals of I_1 that contains infinitely many terms of the sequence $\{x_n\}$. Choose an element $x_{n_2} \in I_2$ with $n_2 > n_1$. Continuing the process, we obtain nested closed intervals $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$ and subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \in I_k$ for all $k \geqslant 1$. By the nested interval theorem, $\exists\ \alpha \in \bigcap_{k=1}^{\infty} I_k$. Clearly, $\alpha \in [a, b]$. Since $\alpha, x_{n_k} \in I_k$, we have

$$d(\alpha, x_{n_k}) \leqslant (b - a)/2^k$$

Thus we have $x_{n_k} \to \alpha$. □

Theorem 3.5. $\mathbb{R}$ *is a complete metric space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. By Lemma 3.7, it follows that $\{x_n\}$ is a bounded sequence in $\mathbb{R}$. Applying the Bolzano-Weierstrass theorem, we get that there is a subsequence $\{x_{n_k}\}$ and a number α such that $x_{n_k} \to \alpha$. Therefore, we obtain by Lemma 3.9 that $x_n \to \alpha$, i.e. $\{x_n\}$ is convergent. □

Example 3.17 (Space of Bounded Functions)**.** Let Ω be a non-empty set. Define the set

$$B(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is bounded}\}.$$

Here a function "$f \in B(\Omega)$ is bounded" means there is a constant $M > 0$ such that

$$|f(x)| \leqslant M \quad \text{for all } x \in \Omega.$$

For each $f \in B(\Omega)$, we define

$$\|f\| = \sup\{|f(x) : x \in \Omega\}.$$

and put

$$d(f, g) = \|f - g\| \quad \text{for } f, g \in B(\Omega).$$

1. Show that $(B(\Omega), d)$ is a metric space.
2. Prove that $B(\Omega)$ is a complete metric space.

Solution. A presentation. ■

Exercise 3.2.

1. *Use the squeeze theorem and the trick in Example 3.12 to prove the following convergences in* $\mathbb{R}$.
   (a) $\lim_{n\to\infty} n^{1/n} = 1$.
   (b) $\lim_{n\to\infty} na^n = 0$ *for* $a \in (0, 1)$.
2. *Let* $x_n < r$ *for all* $n \in \mathbb{N}$ *and* $x_n \to r$ *in* $\mathbb{R}$. *Prove that* $\{x_n\}$ *has a strictly increasing subsequence.*
3. *Let* $x_1 = 1$ *and* $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$ *for* $n > 0$.
   (a) *Show that* $\sqrt{2} \leqslant x_n \leqslant 2$ *for all* $n \geqslant 2$.
   (b) *Show that* $\{x_n\}$ *is a monotone sequence.*
   (c) *Prove that* $\{x_n\}$ *is convergent and find the limit.*
4. *Let* $\{x_n\}$ *be a sequence in* $\mathbb{R}$ *defined by* $x_1 = 1$, $x_{n+1} = \dfrac{x_n + 2}{x_n + 1}$ *for all* $n \in \mathbb{N}$. *Prove that* $\{x_n\}$ *converges and determine the limit. (Hint:* $\{x_{2k}\}$ *and* $\{x_{2k+1}\}$ *are monotonic.)*
5. *Prove that every Cauchy sequence in a metric space is bounded.*
6. *In the metric space* $\mathbb{Q}$, *consider the sequence defined in Example 3.14. Show that this is a Cauchy sequence but it does not converge in* $\mathbb{Q}$.
7. *Let* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *be bounded sequence of real numbers. Prove that there are subsequences* $\{x_{n_k}\}_{k=1}^{\infty}$ *and* $\{y_{n_k}\}_{k=1}^{\infty}$ *(with the same indices) that converge.*
8. *Let* X *be a metric space and* $Y \subset X$ *a subspace of* X.
   (a) *Prove that if* X *is complete and* Y *is closed, then* Y *is complete.*
   (b) *Prove that if* Y *is complete, then* Y *is closed in* X.

Appendix($\star$)

Theorem 3.6. $\mathbb{R}^n$ *is a complete metric space.*

Proof. Same as the proof of the Bolzano-Weierstrass theorem, but employ partitioning cubes. □

Theorem 3.7. *Every compact metric space is complete.*

Proof. Let X be a compact metric space and $\{x_n\}$ a Cauchy sequence in X. To show that $\{x_n\}$ is convergent, we employ Lemma 3.9, so it suffices to show that $\{x_n\}$ has a convergent subsequence. We assume X has infinitely many elements, otherwise, the desired result is trivial.

We perform the following construction. We denote $X_1 := X$, which is a compact metric space. Define $\mathcal{B}_1$ to be the collection of all balls of radius 1 in X_1. Then $\mathcal{B}_1$ is an open cover of X_1. Since X_1 is compact, $\mathcal{B}_1$ contains a finite sub-cover of X_1. In particular, there is $B_1 \in \mathcal{B}_1$ that has infinitely many terms of $\{x_n\}$. Choose a term of the sequence that lies in B_1 and call it x_{n_1}.

Next, we denote $X_2 := \bar{B}_1$–the closed ball of B_1. Since $\bar{B}_1$ is a closed set in X_1 and X_1 is compact, X_2 is compact. Define $\mathcal{B}_2$ to be the collection of all balls of radius $1/2$ in X_2. Then $\mathcal{B}_2$ is an open cover of X_2. Since X_2 is compact, $\mathcal{B}_2$ contains a finite sub-cover of X_2. In particular, there is $B_2 \in \mathcal{B}_2$ that has infinitely many terms of $\{x_n\}$. Choose a term of the sequence that lies in $B_2 \setminus \{x_{n_1}\}$ and call it x_{n_2}. Observe that $B_1 \supset B_2$.

Continuing the process, we get balls

$$B_1 \supset B_2 \supset \cdots \supset B_k \supset \cdots,$$

where B_k has radius $1/k$, and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \in B_k \setminus \{x_{n_1}, \ldots, x_{n_{k-1}}\} \quad \text{for all } k.$$

Lemma 3.10 (Nested compact subsets theorem). *Let $K_1, K_2, \ldots$ be non-empty compact subsets of a metric space X such that $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$. Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.*

Proof. We prove by contradiction. Suppose $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Let $U_n = X \setminus K_n$. Then $\{U_n\}$ is an open cover of X because $\bigcup_{n=1}^{\infty} U_n = X \setminus \bigcap_{n=1}^{\infty} K_n = X$. So $\{U_n\}$ is an open cover of K_1. Since K_1 is compact, $\exists$ a finite sub-collection $\{U_{n_1}, \ldots, U_{n_k}\}$ that covers K_1. So $K_1 \subset U_{n_1} \cap \cdots \cap U_{n_k}$. This implies $K_1 \cap K_{n_1} = \emptyset$, a contradiction. $\square$

Now employing the preceding lemma, we get an element $a \in \bigcap_{n=1}^{\infty} \bar{B}_n$. It can be shown directly that $x_{n_k} \to a$. $\square$

Mathematical Analysis: Lecture 9

Sujin Khomrutai, Ph.D.

Summary. *In this section we study*

1. *∞ and $-\infty$ and extended real number system*
2. *Cluster points (or subsequential limits)*
3. *Limsup and liminf.*

3.4 The extended real number system

In this lecture, we are dealing with special divergent sequences in $\mathbb{R}$. The symbols $\infty, -\infty$ will be used to express the relevant notions. To understand the role of the two symbols, we begin with the following standard definition.

Definition 3.6. The **extended real number system**, denoted by $\bar{\mathbb{R}}$, is the set of all real numbers together with two symbols ∞ and $-\infty$ ($\infty \neq -\infty$), that is

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\},$$

where, in the order structure of $\mathbb{R}$, it is imposed with an order property that

$$-\infty < \infty \quad \text{and} \quad -\infty < x < \infty \quad \text{for all } x \in \mathbb{R}.$$

Remark.

1. In calculus, ∞ and $-\infty$ are symbols that represent *the behaviors of being arbitrarily large and small*, respectively. For instance, the function $f(x) = 1/x$ is arbitrary large when $x \to 0$ and $x > 0$ and is arbitrary small when $x \to 0$ and $x < 0$. Then one puts

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

2. $\infty - \infty$, $-\infty + \infty$, $0 \cdot \infty, 0 \cdot (-\infty)$ cannot be defined. So $\bar{\mathbb{R}}$ is neither a field nor a vector space. However, there is a convention to put the following identities:

$$\begin{aligned}
&\infty + \infty = \infty, \quad -\infty - \infty = -\infty \\
&x \in \mathbb{R}: \quad x + \infty = \infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0, \\
&x > 0: \quad x \cdot (\infty) = \infty, \quad x \cdot (-\infty) = -\infty, \\
&x < 0: \quad x \cdot (\infty) = -\infty, \quad x \cdot (-\infty) = \infty.
\end{aligned}$$

One can check that these are in accordance with the consideration of $\infty, -\infty$ as in 1.

Every non-empty subset of $\bar{\mathbb{R}}$ always has ∞ as an upper-bound and $-\infty$ as a lower-bound.

Definition 3.7 (**Infinite sup or inf**)**.** Let $E \subset \mathbb{R}$ be a non-empty set

- If E is unbounded above in $\mathbb{R}$, we put
$$\sup E = \infty.$$
- If $E \subset \mathbb{R}$ is unbounded below in $\mathbb{R}$, we put
$$\inf E = -\infty.$$

Example 3.18. Let $E = \{n + \frac{1}{n} : n \in \mathbb{N}\}$. Find $\sup E$ and $\inf E$.

Solution. Since E is unbounded above, $\sup E = \infty$. Also, $n + \frac{1}{n} \geqslant 2$ (by induction), it follows that $\inf E = 2$. ■

Next, we are considering sequences.

Definition 3.8 (**Infinite limits**)**.** Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

(a) If for every $R > 0$ there is $N \in \mathbb{N}$ such that if $n \geqslant N$ then $x_n > R$, we say that $\{x_n\}$ **diverges to** ∞ and write
$$x_n \to \infty \quad \text{or} \quad \lim_{n\to\infty} x_n = \infty.$$

(b) If for every $R > 0$ there is $N \in \mathbb{N}$ such that if $n \geqslant N$ then $x_n < -R$, we say that $\{x_n\}$ **diverges to** $-\infty$ and write
$$x_n \to -\infty \quad \text{or} \quad \lim_{n\to\infty} x_n = -\infty.$$

Question. Prove that $x_n \to -\infty$ if and only if $-x_n \to \infty$.

Example 3.19. Prove that $\lim_{n\to\infty} \frac{n^2+1}{5n} = \infty$.

Solution. Let $R > 0$. Choose $N \in \mathbb{N}$ such that $N > 5R$. For $n \geqslant N$, we have

$$x_n = \frac{n^2+1}{5n} = \frac{n}{5} + \frac{1}{5n} > R.$$

So $x_n \to \infty$. ∎

Question. Prove that if $x_n \to \infty$ then $\sup\{x_n : n \in \mathbb{N}\} = \infty$. The converse is not true. For $x_n = (-1)^n n$, show that

$$\sup\{x_n : n \in \mathbb{N}\} = \infty,$$

but the sequence $\{x_n\}$ does not converge to ∞.

Lemma 3.11 (Criteria for infinite limits).

1. *$x_n \to \infty$ if and only if $-x_n \to -\infty$.*
2. *$x_n \to \infty$ if and only if $\exists\, M \in \mathbb{N}$ such that $x_n > 0$ for all $n \geqslant M$ and $\frac{1}{x_n} \to 0$.*
3. *$x_n \to -\infty$ if and only if $\exists\, M \in \mathbb{N}$ such that $x_n < 0$ for all $n \geqslant M$ and $\frac{1}{x_n} \to 0$.*

Proof. 1. Done.

2. ($\Rightarrow$) Suppose $x_n \to \infty$. Let $\varepsilon > 0$. Choose $R > 0$ such that $1/R < \varepsilon$. By Definition 3.8, $\exists\, N \in \mathbb{N}$ such that $x_n > R$ for all $n \geqslant N$, in particular, $x_n > 0$. Now if $n \geqslant N$, then $|1/x_n| = 1/x_n < 1/R < \varepsilon$. Thus $1/x_n \to 0$.

($\Leftarrow$) Suppose $x_n > 0$ for $n \geqslant M$ and $1/x_n \to 0$. We have to show that $x_n \to \infty$. Let $R > 0$. Since $1/x_n \to 0$, $\exists\, N \in \mathbb{N}$ such that if $n \geqslant N$ then $|1/x_n| < 1/R$. Now if $n \geqslant \max\{N, M\}$ then $x_n = |x_n| > R$. Thus $x_n \to \infty$.

3. This follows from 1 and 2 by considering the sequence $\{-x_n\}$. □

Example 3.20. Show that $x_n = \frac{2n-n^2}{3n+4} \to -\infty$.

Solution. For $n \geqslant 3$, it is clearly that $x_n < 0$. Consider

$$\frac{1}{x_n} = \frac{3n+4}{2n-n^2} = \frac{\frac{3}{n}+\frac{4}{n^2}}{\frac{2}{n}-1} \to 0.$$

So $x_n \to -\infty$. ∎

Example 3.21. Let $x_1 = a > 0$ and $x_{n+1} = 2x_n + \frac{1}{x_n}$. Prove that $x_n \to \infty$.

Solution. Clearly, $x_n > 0$ for all n. Let $y_n = 1/x_n$. Then $y_1 = 1/a$ and

$$y_{n+1} = \frac{y_n}{2+y_n^2} \leqslant \frac{y_n}{2}.$$

By induction, one can show that $y_{n+k} \leqslant (1/2^k)y_n$. This implies $y_{k+1} \leqslant 1/(2^k a) \to 0$ as $k \to \infty$. So $x_n \to \infty$. ∎

3.5 Limsup and liminf

In this section, we consider sequences in a metric space X. For sequences in $\mathbb{R}$, we shall assume they are bounded. In the context of $\mathbb{R}$, it is possible, as in some textbooks, to include unbounded sequences into consideration and employ $\infty, -\infty$ for unbounded above and below, respectively. See the remark below.

Definition 3.9 (**Cluster points**)**.** Let $\{x_n\}$ be a sequence in a metric space (X, d). An element $b \in X$ is called a **cluster point** (or a **subsequential limit**) of $\{x_n\}$ if there is a subsequence $x_{n_k} \to b$ in X.

Example 3.22.

1. Consider the sequence $\{x_n = 1 - (-1)^n\}$ in $\mathbb{R}$. We have $x_{2k} = 0 \to 0$ and $x_{2k-1} = 2 \to 2$. So $0, 2$ are cluster points of $\{x_n\}$.
2. If $x_n \to \infty$ or $x_n \to -\infty$, then there is no cluster point for $\{x_n\}$.

■

Question. Show that for the sequence $\{x_n = 1 - (-1)^n\}$, 0 and 2 are the only cluster points. See also the exercise.

Remark. For a sequence $\{x_n\}$ in $\mathbb{R}$, if $\{x_n\}$ is unbounded from above, we say that ∞ is a cluster point of the sequence. Similarly, we say that $-\infty$ is a cluster point of $\{x_n\}$ if it is unbounded from below. In this way, it follows that every sequence in $\mathbb{R}$ always has a limit point in $\bar{\mathbb{R}}$.

Example 3.23 $(*)$**.** Since $\mathbb{Q}$ is a countable set, there is a bijection $\mathbb{N} \to \mathbb{Q}$. We can denote such a bijection by a sequence $\{r_n\}_{n=1}^{\infty}$, called an **enumeration of** $\mathbb{Q}$.

Prove that every real number x_0 is a subsequential limit of the sequence $\{r_n\}_{n=1}^{\infty}$.

Solution. First note that, for each $k \in \mathbb{N}$, the interval $(x_0 - 1/k, x_0 + 1/k)$ contains infinitely many terms of $\{r_n\}_{n=1}^{\infty}$.

Now for $k = 1$, we choose n_1 such that $r_{n_1} \in (x_0 - 1, x_0 + 1)$, for $k = 2$, we can choose $n_2 > n_1$ such that $r_{n_2} \in (x_0 - 1/2, x_0 + 1/2)$, and so on. Thus we get a subsequence $\{r_{n_k}\}_{k=1}^{\infty}$ such that $x_0 - 1/k < r_{n_k} < x_0 + 1/k$, that is

$$|r_{n_k} - x_0| < 1/k \quad \text{for all } k \in \mathbb{N}.$$

By the squeeze theorem (Theorem 3.2), we get that $\lim_{k\to\infty} |r_{n_k} - x_0| = 0$, hence $r_{n_k} \to x$. ■

Lemma 3.12. *A sequence $\{x_n\}$ is convergent in X if and only if $\{x_n\}$ has exactly one cluster point.*

Proof. Exercise. □

By the Bolzano-Weierstrass theorem, every bounded sequence in $\mathbb{R}$ has a cluster point.

Definition 3.10 (Limsup and liminf). Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. We consider the set of all cluster points of $\{x_n\}$:

$$E = \{b \in \mathbb{R} : \exists\, x_{n_k} \to b\} \quad (E \neq \emptyset, \text{by the Bolzano-Weierstras theorem}).$$

(a) The **limit supremum** of $\{x_n\}$ is the number denoted by

$$\limsup_{n\to\infty} x_n = \sup E.$$

(b) The **limit infimum** of $\{x_n\}$ is the number given by

$$\liminf_{n\to\infty} x_n = \inf E.$$

If $L \leqslant x_n \leqslant M$ for all $n \in \mathbb{N}$, then

$$L \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant M.$$

Example 3.24. Let $x_n = (-1)^n \dfrac{n}{n+1}$. Find $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$.

Solution. $\{x_n\}$ is partitioned into two convergent subsequences:

$$x_{2k} = \frac{2k}{2k+1} \to 1, \quad x_{2k+1} = -\frac{2k+1}{2k+2} \to -1.$$

So $E = \{-1, 1\}$, hence $\limsup\limits_{n\to\infty} x_n = 1$ and $\liminf\limits_{n\to\infty} x_n = -1$. ■

Theorem 3.8. *Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$.*

1. $\limsup\limits_{n\to\infty} x_n = x^* \Leftrightarrow \forall\, \varepsilon > 0$, *$\exists$ finitely many n such that $x_n \geqslant x^* + \varepsilon$ and $\exists$ infinitely many n such that $x^* - \varepsilon < x_n < x^* + \varepsilon$.*
2. $\liminf\limits_{n\to\infty} x_n = x_* \Leftrightarrow \forall\, \varepsilon > 0$, *$\exists$ finitely many n such that $x_n \leqslant x_* - \varepsilon$ and $\exists$ infinitely many n such that $x_* - \varepsilon < x_n < x_* + \varepsilon$.*

Proof. Prove by contradiction. See Exercise. □

Corollary 3.1. *For a bounded sequence $\{x_n\}$ in $\mathbb{R}$, $x^* = \limsup\limits_{n\to\infty} x_n$ and $x_* = \liminf\limits_{n\to\infty} x_n$ are cluster points and any other cluster point x' satisfies $x_* \leqslant x' \leqslant x^*$.*

Proof. Follows from the definition and the preceding theorem. □

Theorem 3.9. *Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then*

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geqslant n\}$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geqslant n\}.$$

Proof. We only prove the first identity, the second one is left as an exercise.

Let $a = \limsup_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} \sup\{x_k : k \geqslant n\}$. Since $\{x_n\}$ is bounded, both $a, b \in \mathbb{R}$.

Let $\varepsilon > 0$. By Theorem 3.8, there are finitely many x_n such that $x_n \geqslant a + \varepsilon$. So $b \leqslant a + \varepsilon$ (why?). This is true for all $\varepsilon > 0$, hence $b \leqslant a$.

Since the sequence $\sup\{x_k : k \geqslant n\}$ is decreasing to b, there are only finitely many n such that $x_n \geqslant b_n + \varepsilon$. This implies $a \leqslant b + \varepsilon$ (why?). Analogously, we have $a \leqslant b$. Therefore $a = b$. □

Corollary 3.2. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Then*

1. $\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$
2. $\limsup_{n\to\infty}(x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$
3. $\liminf_{n\to\infty}(x_n + y_n) \geqslant \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$
4. *If $c \geqslant 0$ then $\limsup_{n\to\infty} cx_n = c\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} cx_n = c\liminf_{n\to\infty} x_n$*
5. *If $x_n \geqslant 0, y_n \geqslant 0$ for all n then*
$$\limsup_{n\to\infty} x_n y_n \leqslant \limsup_{n\to\infty} x_n \limsup_{n\to\infty} y_n, \quad \liminf_{n\to\infty} x_n y_n \leqslant \liminf_{n\to\infty} x_n \liminf_{n\to\infty} y_n$$
6. *If $x_n \leqslant y_n$ for all n then*
$$\limsup_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} y_n, \quad \liminf_{n\to\infty} x_n \leqslant \liminf_{n\to\infty} y_n.$$

Proof. Exercise. □

Remark. If $\{x_n\}$ is unbounded above, we put

$$\limsup_{n\to\infty} x_n = \infty,$$

and if $\{x_n\}$ is unbounded from below, we put

$$\liminf_{n\to\infty} x_n = -\infty.$$

Exercise 3.3.

1. *Prove Theorem 3.8.*

2. *Prove Theorem 3.9.*

3. *Prove that if* $\{x_n\}$ *is split into two subsequences* $\{x_{n_k}\}$, $\{x_{n_l}\}$, *i.e.* $\{n_k : k \in \mathbb{N}\}$ *and* $\{n_l : l \in \mathbb{N}\}$ *form a partition of* $\mathbb{N}$, *and* $x_{n_k} \to a$, $x_{n_l} \to b$ *then* a, b *are the only sequential limits of* $\{x_n\}$. *Can this statement be generalized to arbitrary number of subsequences?*

4. *Find limsup and liminf for each of the following sequences in* $\mathbb{R}$.

   (a) $x_n = \dfrac{(-1)^n 5n + 7}{3n + 5}$

   (b) $y_n = n^{\sin(n\pi/2)} + (1/n)\cos n$

5. *For any two sequences* $\{x_n\}, \{y_n\}$ *in* $\mathbb{R}$, *prove that*

$$\limsup_{n\to\infty}(x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

6. *Let* $x_n \geqslant 0$ *and* $y_n \to b$ *with* $b > 0$. *Prove that*

$$\limsup_{n\to\infty} x_n y_n = b \limsup_{n\to\infty} x_n, \quad \liminf_{n\to\infty} x_n y_n = b \liminf_{n\to\infty} x_n.$$

Mathematical Analysis: Lecture 10

Sujin Khomrutai, Ph.D.

Summary. *In this section we study*

1. *Series of real numbers and the Cauchy criterion,*
2. *Comparison, ratio, and root tests,*
3. *Absolutely convergence vs conditional convergence*

3.6 Series of Real Numbers

Definition 3.11 (Infinite series). Let $\{x_n\}$ be a sequence in $\mathbb{R}$. We form the sequence $\{s_n\}$ in $\mathbb{R}$ by

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \quad (n \in \mathbb{N}).$$

$\{s_n\}$ is called an **infinite series** (or simply a **series**) and will be denoted by

$$\sum_{n=1}^{\infty} x_n \quad \text{or} \quad \sum x_n.$$

Each s_n is called the n'th **partial sum** of the series. If $s_n \to s$ we say that the series $\sum x_n$ **converges**, s is called the **sum of the series**, and denote

$$s = \sum_{n=1}^{\infty} x_n.$$

If $\{s_n\}$ diverges, we say that the series $\sum x_n$ is **divergent**.

Example 3.25. Determine the convergence of the series $\sum_{n=1}^{\infty} 1/2^n$ and $\sum_{n=1}^{\infty} n$.

Solution. For the first series, the n'th partial sum is

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Since $1 - 1/2^n \to 1$, the first series converges and the sum is $\sum_{n=1}^{\infty} 1/2^n = 1$.

For the second series, the n'th partial sum is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \to \infty$$

so the second series diverges with $\sum_{n=1}^{\infty} n = +\infty$. ■

Some known series.

1. (geometric series) $\sum ar^{n-1}$ converges when $-1 < r < 1$ and the sum is $a/(1-r)$.
2. (harmonic series) $\sum 1/n$ diverges. See below.
3. (p-series) $\sum 1/n^p$ converges when $p > 1$ and diverges otherwise. See Example 3.31.

Theorem 3.10. *Let $\sum x_n = L$ and $\sum y_n = M$. Then for any constants $a, b \in \mathbb{R}$, $\sum(ax_n + by_n) = aL + bM$.*

Proof. This follows from that the n'th partial sum of $\sum(ax_n + by_n)$ satisfies

$$\sum_{k=1}^{n}(ax_k + by_k) = a\sum_{k=1}^{n} x_k + b\sum_{k=1}^{n} y_k \to aL + bM.$$

□

Example 3.26. Evaluate the series $\sum \left(\frac{5 \cdot 3^n}{4^n} + \frac{(-2)^{2n-1}}{5^n}\right)$.

Solution. The series is $5\sum(\frac{3}{4})^n + \frac{1}{2}\sum(-\frac{4}{5})^n = 5\frac{3/4}{1-3/4} + \frac{1}{2}\frac{-4/5}{1-(-4/5)} = 15 - \frac{2}{9}$. ■

Theorem 3.11 (Cauchy criterion). *A series $\sum_{n=1}^{\infty} x_n$ converges if and only if for every $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that*

$$n \geqslant m \geqslant N \quad \Rightarrow \quad \left|\sum_{k=m}^{n} x_k\right| \leqslant \varepsilon.$$

Proof. Since $\mathbb{R}$ is complete, the series converges if and only if the sequence of partial sums is Cauchy. Now

$$s_n - s_{m-1} = \sum_{k=m}^{n} x_k,$$

so the desired statement is true. □

We present some basic applications of the Cauchy criterion.

Corollary 3.3. *If* $\sum x_n$ *converges then* $\lim_{n\to\infty} x_n = 0$.

Proof. We have $x_n = s_n - s_{n-1} \to s - s = 0$. □

The converse of the above corollary is generally false.

Question. For $x_n = 1/n$, show that $x_n \to 0$ but $\sum x_n$ diverges.

Example 3.27. Let x_n be a decreasing sequence of real numbers such that $x_n \to 0$. Suppose the series $\sum x_n$ converges. Prove that

$$\lim_{n\to\infty} n x_n = 0.$$

Solution. Let $\varepsilon > 0$. By the Cauchy criterion (Theorem 3.11), $\exists\, M \in \mathbb{N}$ such that if $n \geqslant m \geqslant M$ then

$$\sum_{k=m}^{n} x_k < \frac{\varepsilon}{2}.$$

Let $N = 2M$. For $n \geqslant N$, since $\{x_n\}$ is decreasing, we have

$$\frac{n}{2} x_n \leqslant x_N + x_{N+1} + \cdots + x_n < \frac{\varepsilon}{2}.$$

This implies $n x_n < \varepsilon$. Thus $n x_n \to 0$. ■

Remark. The preceding example is sharp in the sense that there is a series $\sum x_n$ such that $\{x_n\}$ is decreasing, $x_n \to 0$, $\sum x_n$ converges, but

$$\lim_{n\to\infty} n^{1+\varepsilon} x_n > 0.$$

Theorem 3.12 (**Comparison test**).

1. *Suppose* $|x_n| \leqslant a_n$ *for all* $n \geqslant n_0$ *(*n_0 *is fixed) and* $\sum a_n$ *converges, then* $\sum x_n$ *converges.*
2. *If* $x_n \geqslant b_n$ *for all* $n \geqslant n_0$ *(*n_0 *is fixed) and* $\sum b_n$ *diverges, then* $\sum x_n$ *diverges.*

Proof. 1. We apply the Cauchy criterion (Theorem 3.11). Let $\varepsilon > 0$. Since $\sum a_n$ converges, $\exists\, M \in \mathbb{N}$ such that if $n \geqslant m \geqslant M$ then

$$\sum_{k=m}^{n} a_k \leqslant \varepsilon.$$

We take $N = \max\{n_0, M\}$. If $n \geqslant m \geqslant N$, we have by the triangle inequality that

$$\left|\sum_{k=m}^{n} x_k\right| \leqslant \sum_{k=m}^{n} |x_k| \leqslant \sum_{k=m}^{n} a_k < \varepsilon.$$

So $\sum x_n$ converges.

2. That $b_n \geqslant 0$ and $\sum b_n$ diverges implies the sequence of partial sums $\sum_{k=1}^{n} b_k \to \infty$. So the sequence of partial sums of $\sum x_n$, which is $\geqslant$ the sequence of partial sums of $\sum b_n$, must diverges as well. □

Corollary 3.4. *If $\sum |x_n|$ converges then $\sum x_n$ converges.*

Proof. Use the comparison test with $a_n = |x_n|$. □

Example 3.28. Show that $\sum \dfrac{(-1)^n \sqrt{n^2-5}}{4+n^3}$ converges.

Solution. We have $|x_n| = \dfrac{\sqrt{n^2-5}}{4+n^3} \leqslant \dfrac{n}{n^3} = \dfrac{1}{n^2}$ and $\sum 1/n^2$ converges, so $\sum x_n$ converges by the comparison test. ■

Example 3.29. Let $x_n > 0$ for all $n \in \mathbb{N}$ and $\sum x_n$ diverges. Prove that

$$\sum \frac{x_n}{1+x_n} \quad \text{diverges.}$$

Solution. If x_n is bounded, $\exists\, M > 0$ such that $x_n \leqslant M$ for all $n \in \mathbb{N}$. Then

$$\frac{x_n}{1+x_n} \geqslant \frac{1}{1+M} x_n.$$

Since $\sum \dfrac{1}{1+M} x_n$ diverges, $\sum \dfrac{x_n}{1+x_n}$ diverges, by the comparison test.

Now assume x_n is not bounded. So there is a subsequence $x_{n_k} \to \infty$. We can assume $x_{n_k} \geqslant 1$ for all $k \in \mathbb{N}$. Note that $x/(1+x) \geqslant 1/2$ for all $x \geqslant 1$. Then

$$\frac{x_{n_k}}{1+x_{n_k}} \geqslant \frac{1}{2} \quad \forall\, k \in \mathbb{N}.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{1+x_n} \geqslant \sum_{k=1}^{\infty} \frac{x_{n_k}}{1+x_{n_k}}$ diverges. ■

Example 3.30. Let $\sum x_n$ be a convergent series with $x_n \geqslant 0$. Prove that

$$\sum \frac{\sqrt{x_n}}{n} \quad \text{is convergent.}$$

Solution. We have

$$\frac{\sqrt{x_n}}{n} \leqslant \frac{1}{2}\left(x_n + \frac{1}{n^2}\right) \quad \forall\, n \in \mathbb{N}.$$

Since $\sum x_n$ and $\sum 1/n^2$ are convergent, we get $\sum \sqrt{x_n}/n$ is convergent too. ■

Definition 3.12. A series $\sum x_n$ is said to be **absolutely convergent** provided $\sum |x_n|$ converges. By Corollary 3.4, every absolutely convergent series converges.

If $\sum |x_n|$ diverges but $\sum x_n$ converges, $\sum x_n$ is said to be **conditionally convergent**.

Question. Let $x_n \geqslant 0$ and $s_n = x_1 + \cdots + x_n$. Show that the series $\sum x_n$ converges if and only if $\{s_n\}$ is a bounded sequence.

Show that the series $\sum x_n$ converges if and only if $\{s_n\}$ has a convergent subsequence.

Theorem 3.13 (**Test for series with decreasing terms**). *Let $\{x_n\}$ be a decreasing sequence with $x_n \geqslant 0$ for all $n \in \mathbb{N}$. We have $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k x_{2^k}$ converges.*

Proof. $\sum x_n$ and $\sum 2^k x_{2^k}$, respectively, have the sequences of partial sums

$$s_n = x_1 + x_2 + \cdots + x_n \quad (n = 1, 2, \ldots),$$
$$u_k = x_1 + 2x_2 + 4x_4 + \cdots + 2^k x_{2^k} \quad (k = 0, 1, 2, \ldots).$$

Since $\{x_n\}$ is decreasing, we have $x_{2^k} + x_{2^k+1} + \cdots + x_{2^{k+1}-1} \leqslant 2^k x_{2^k}$, which implies

$$x_1 + (x_2 + x_3) + \cdots + (x_{2^k} + \cdots + x_{2^{k+1}-1}) \leqslant x_1 + 2x_2 + \cdots + 2^k x_{2^k}.$$

This means

$$s_{2^{k+1}-1} \leqslant u_k \quad \text{for } k = 0, 1, 2 \ldots.$$

Similarly, since $x_{2^{k-1}+1} + x_{2^{k-1}+2} + \cdots + x_{2^k} \geqslant 2^{k-1} x_{2^k}$, it follows that

$$s_{2^k} \geqslant \frac{1}{2} u_k \quad (k = 0, 1, 2, \ldots).$$

($\Rightarrow$) Suppose $\sum x_n$ is convergent. Then $\{s_n\}$ converges, so does the subsequence $\{s_{2^k}\}$. Thus $\{u_k\}$ converges, which means $\sum 2^k x_{2^k}$ is convergent.

($\Leftarrow$) Suppose $\sum 2^k x_{2^k}$ is convergent. Then $\{u_k\}$ converges, so $\{s_{2^{k+1}-1}\}$ is a convergent subsequence of $\{s_n\}$. Hence $\{s_n\}$ converges. □

Example 3.31. Show that the p-series $\sum 1/n^p$ converges if and only if $p > 1$.

Solution. We only consider $p > 0$. The terms $x_n = 1/n^p$ are decreasing. Consider $2^k x_{2^k} = 2^k/2^{kp} = 2^{k(1-p)}$. So $\sum 2^k x_{2^k}$ converges if and only if $p > 1$. ■

Theorem 3.14 (**Root test and ratio test**). *Let $\sum x_n$ be a series of real numbers and $L = \limsup_{n\to\infty} |x_n|^{1/n}$ and $R = \limsup_{n\to\infty} |x_{n+1}|/|x_n|$.*

(1) If $L < 1$ or $R < 1$, then $\sum x_n$ is absolutely convergent.

(2) If $L > 1$ or $R > 1$, then $\sum x_n$ diverges.

(3) If $L = 1$ the root test gives no information.

Proof. Left as an exercise. □

Example 3.32. Determine whether the convergence or divergence of the following series.

- $\sum \frac{n^n}{4^{1+6n}}$
- $\sum \left(\frac{5n - 3n^2}{8n^2 + n}\right)^n$

Solution. By the ratio test, the first series is divergent, while the second one is convergent. ■

Exercise 3.4.

1. *If $\sum x_n$ converges and $\{a_n\}$ is a bounded monotone sequence, prove that $\sum a_n x_n$ converges. Is it still true when $\{a_n\}$ is only bounded?*
2. *Does the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ converges?*
3. *Assume $x_n \neq 0$ for all $n \geqslant n_0$. Show that*

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty} |x_n|^{1/n} \leqslant \limsup_{n\to\infty} |x_n|^{1/n} \leqslant \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}.$$

Hint. Use Theorem 3.8 and that $(a + \varepsilon/n)^n \geqslant a + \varepsilon$ $(a \geqslant 0, \varepsilon > 0)$.

Note. This result and the series $\sum 2^{-n+a_n}$ ($a_n = 1$ if n even, $a_n = 0$ if n odd) implies that the root test is stronger than the ratio test.

4. *Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s}$ diverges if $s = 1$ and converges if $s > 1$.*
5. *Prove the **summation by parts formula**:*

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} s_k(b_k - b_{k+1}) + s_n b_n - s_{m-1} b_m,$$

where $s_0 = 0, s_n = a_1 + \cdots + a_n$ $(n \in \mathbb{N})$.

6. *Use the summation by parts formula to prove the convergence of $\sum (-1)^{n+1}/n$.*

■

Mathematical Analysis: Lecture 11

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we study*

1. *Limit points of a subset of metric space,*
2. *Limit of functions, the sequntial characterization, and some properties,*
3. *Limits involving infinities, limsup and liminf.*

4 Limits and Continuity of Functions

We study the concepts of limits and continuity of functions $f : X \to Y$, where X, Y are metric spaces. A central result is that concepts developed in this chapter can be expressed in terms of sequences. We will also explore the connection of continuity to connectedness and compactness.

Throughout this chapter, (X, d) and (Y, ρ) are metric spaces. Our primary interest will be $X = Y = \mathbb{R}$.

4.1 Limit points and limits of functions

From calculus, in justifying the limit behavior of a real-valued function f as x approaches a real number x_0, there must be an $x \neq x_0$, in every distance $r > 0$ from x_0, that lies in the domain of f. This motivates the following definition.

Definition 4.1 (Limit points). Let $E \subset X$ be a non-empty set.

1. $x_0 \in X$ is called a **limit point** (or **accumulation point**) of E if $\forall\, r > 0$, $(E \cap B_r(x_0)) \setminus \{x_0\} \neq \emptyset$. A limit point may or may not be an element of E.
2. $x_0 \in X$ is called an **isolated point** of E if $\exists\, r > 0$, $E \cap B_r(x_0) = \{x_0\}$. In particular, an isolated point is not a limit point.

Example 4.1. Let $E = (0, 1] \cup \{3\}$, a subset of $\mathbb{R}$. Then the set of all limit points of E is $[0, 1]$. E has 3 is the only isolated point. Note that 0 which is a limit point of E is not an element of E. ■

Example 4.2. Let X be a non-empty set with the discrete metric and $E \subset X$ a non-empty set. Since for any $x_0 \in X$, $B_{1/2}(x_0) = \{x_0\}$, we have

$$E \cap B_{1/2}(x_0) = \{x_0\}.$$

So E has no limit point and every point of E is isolated. ■

Example 4.3. Consider an open ball $B_r(\vec{x}_0)$ in $\mathbb{R}^2$. Every $\vec{x} \in B_r(\vec{x}_0)$ is a limit point by the triangle inequality. There is no isolated point for this set. Furthermore, every $\vec{x}$ with $\|\vec{x} - \vec{x}_0\| = r$ is a limit point not lying in the set. ■

Lemma 4.1 (**Limit points as limit of sequences**). *A point x_0 is a limit point of E if and only if there is a sequence $\{x_n\}$ in $E \setminus \{x_0\}$ such that $x_n \to x_0$ in X. In particular, if E has a limit point then E is an infinite set.*

Question. Prove the above lemma.

Example 4.4. Every finite set has no limit point. ■

From now one, unless stated otherwise, assume

E is a non-empty subset of X and x_0 is a limit point of E.

Our prime example is E is an interval in $\mathbb{R}$ and $x_0 \in E$ or at one of the endpoints.

Definition 4.2 (**Limit of functions**). Let $f : E \subset X \to Y$ be a function. We say that $f(x)$ **converges to** L as $x \to x_0$ in E if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ such that if } x \in E \setminus \{x_0\},\ d(x, x_0) < \delta \text{ then } \rho(f(x), L) < \varepsilon.$$

If $f(x)$ converges to L as $x \to x_0$, we write

$$f(x) \to L \text{ as } x \to x_0, \quad \text{or} \quad \lim_{x \to x_0} f(x) = L.$$

$L \in Y$ (shown to be unique) is called the **limit of** f as x approaches x_0.

If there is no such $L \in Y$, we say that $\lim_{x \to x_0} f(x)$ **does not exist**.

Remark.

1. The inequality $\rho(f(x), L) < \varepsilon$ can be replaced with $\rho(f(x), L) < K\varepsilon$, where $K > 0$ is a constant (may depend on x_0, though).

2. If $f : X \to \mathbb{R}$, that $\lim_{x \to x_0} f(x) = L$ means

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ such that if } x \in E \setminus \{x_0\},\ d(x, x_0) < \delta \text{ then } |f(x) - L| < \varepsilon.$$

3. If $X = \mathbb{R}$ one can define the *left-hand* and *right-hand limits*. If $X = Y = \mathbb{R}$ one can define $f(x) \to \pm\infty$ accordingly. See below.

Remark. The definition $f(x) \to L$ as $x \to x_0$ may also be understood as "one can make $f(x)$ *arbitrarily* close to L by taking x *sufficiently* close to x_0". Here by arbitrarily close represents that

$$\text{“}\rho(f(x), L) < \varepsilon \text{ for any } \varepsilon > 0\text{”},$$

and x sufficiently close to x_0 represents

$$\text{“}x \in E \setminus \{x_0\},\ d(x, x_0) < \delta \text{ for a certain } \delta > 0\text{”}.$$

Lemma 4.2 (Uniqueness of limits). *If $f(x) \to L$ and $f(x) \to M$ as $x \to x_0$, then $L = M$.*

Proof. Let $\varepsilon > 0$. That $f(x) \to L$ means $\exists\, \delta_1 > 0$ such that if $x \in E \setminus \{x_0\}$, $d(x, x_0) < \delta_1$, then

$$\rho(f(x), L) < \varepsilon/2.$$

Similarly, $\exists\, \delta_2 > 0$ such that if $x \in E \setminus \{x_0\}$, $d(x, x_0) < \delta_2$ then

$$\rho(f(x), M) < \varepsilon/2.$$

For $x \in E \setminus \{x_0\}$, $d(x, x_0) < \min\{\delta_1, \delta_2\}$, then

$$\rho(L, M) \leqslant \rho(L, f(x)) + \rho(f(x), M) < \varepsilon,$$

by the triangle inequality. This is true for all $\varepsilon > 0$, so $L = M$. □

Example 4.5. Show that $\lim_{x \to x_0} x^2 = x_0^2$.

Solution. Let $\varepsilon > 0$. We choose $\delta = \min\{1, \varepsilon\}$. For $0 < |x - x_0| < \delta$, then

$$|x^2 - x_0^2| = |x + x_0||x - x_0| < (2|x_0| + |x - x_0|)|x - x_0| < (2|x_0| + 1)\varepsilon.$$

It follows that $\lim_{x \to x_0} x^2 = x_0^2$. ■

Example 4.6 (∗). If $x_0 \neq 1/2$, show that

$$\lim_{x \to a} \frac{3x + 2}{2x - 1} = \frac{3x_0 + 2}{2x_0 - 1}.$$

Solution. Here $X = Y = \mathbb{R}$ and $E = \mathbb{R} \setminus \{1/2\}$. Let $f(x) = (3x + 2)/(2x - 1)$. We have to show that $f(x) \to f(x_0)$ as $x \to x_0$, for $x_0 \neq 1/2$. Observe that

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \frac{3x + 2}{2x - 1} - \frac{3x_0 + 2}{2x_0 - 1} \right| \\ &= \left(\frac{7}{|2x - 1||2x_0 - 1|} \right) |x - x_0|. \end{aligned}$$

We require firstly $|x - x_0| < |2x_0 - 1|/4$. Then by the triangle inequality, we have

$$|2x - 1| = |(2x_0 - 1) - 2(x_0 - x)| \geqslant |2x_0 - 1| - 2|x_0 - x| > \frac{1}{2}|2x_0 - 1|.$$

This gives $|f(x) - f(x_0)| < \frac{14}{|2x_0-1|^2}|x - x_0|$. So in addition, we set

$$\delta = \min\{|2x_0 - 1|/2, \varepsilon\}.$$

If $0 < |x - x_0| < \delta$ then $|f(x) - f(x_0)| < \frac{14}{|2x_0-1|^2}\varepsilon$. Thus $f(x) \to f(x_0)$ as $x \to x_0$. ■

In a metric space, the behavior that $x \to x_0$ can be characterized in terms of sequences $x_n \to x_0$. So we get the following sequential characterization.

Theorem 4.1 (**Sequential characterization of limits**). *Let $f : E \subset X \to Y$. We have $\lim_{x\to x_0} f(x) = L \Leftrightarrow \lim_{n\to\infty} f(x_n) = L$, for any $\{x_n\}$ in $E\setminus\{x_0\}$ with $x_n \to x_0$. In particular, if $f(x)$ converges as $x \to x_0$, then $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(x_n)$, for any $\{x_n\}$ in $E \setminus \{x_0\}$, $x_n \to x_0$.*

Proof. ($\Rightarrow$) Suppose $\lim_{x\to x_0} f(x) = L$. Let $\{x_n\}$ be in $E \setminus \{x_0\}$, $x_n \to x_0$. We have to show that $\rho(f(x_n), L) \to 0$.

Let $\varepsilon > 0$. Since $\lim_{x\to x_0} f(x) = L$, $\exists\, \delta > 0$ such that for $x \in E \setminus \{x_0\}$, $d(x, x_0) < \delta$, we get $\rho(f(x), L) < \varepsilon$. Since $x_n \to x_0$, $\exists\, N \in \mathbb{N}$ such that if $n \geqslant N$ then $0 < d(x_n, x_0) < \delta$. Now, for all $n \geqslant N$ we get $\rho(f(x_n), L) < \varepsilon$. So $\rho(f(x_n), L) \to 0$.

($\Leftarrow$) We prove the contrapositive, i.e. assume the statement $f(x)$ does not converge to L, then we have to find $\{x_n\}$ in $E \setminus \{x_0\}$, $x_n \to x_0$, $f(x_n)$ does not converge to L.

The statement "$f(x)$ does not converge to L" means $\exists\, \varepsilon > 0$, $\forall\, \delta > 0$, there is $x \in E \setminus \{x_0\}$ with $d(x, x_0) < \delta$, $\rho(f(x), L) \geqslant \varepsilon$. Take $\delta = 1/n$, we get an element $x_n \in E \setminus \{x_0\}$ such that $d(x_n, x_0) < 1/n$ and $\rho(f(x_n), L) \geqslant \varepsilon$. So $\{x_n\}$ is in $E \setminus \{x_0\}$, $x_n \to x_0$ and $f(x_n)$ does not converge to L as desired. □

Example 4.7. Let $f(x) = \dfrac{5x^2 - 4x + 6}{1 + x^2 + x^4}$. Find $\lim_{x\to x_0} f(x)$.

Solution. Let $\{x_n\}$ be a sequence in $\mathbb{R} \setminus \{x_0\}$ such that $x_n \to x_0$. Then we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{5x_n^2 - 4x_n + 6}{1 + x_n^2 + x_n^4} = \frac{5x_0^2 - 4x_0 + 6}{1 + x_0^2 + x_n^2},$$

where we have used the theorem on limit of sequences. ■

Example 4.8. Let $f(x) = \sin(1/x)$ for $x \neq 0$. Show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

Solution. In view of Theorem 4.1, it suffices to exhibit two subsequences $\{x_n\}, \{y_n\}$ in $\mathbb{R} \setminus \{0\}$ such that $x_n \to 0, y_n \to 0$ but $f(x_n), f(y_n)$ tend to different limits.

Let $x_n = 1/(n\pi)$ and $y_n = 2/(4n+1)\pi$. Observe that $x_n \to 0$, $y_n \to 0$, and

$$f(x_n) = 0 \to 0, \quad f(y_n) = 1 \to 0.$$

So $\lim_{x\to 0} f(x)$ does not exist.

Alternatively, one observes that $f(2/(2n-1)\pi) = (-1)^n$ does not converge. So $f(x) = \sin(1/x)$ does not converge as $x \to 0$. ∎

Example 4.9. Determine the limit $\displaystyle\lim_{x\to 2} \frac{x + \lfloor x \rfloor}{x - \lfloor x \rfloor}$.

Solution. Consider a sequence $\{x_n\}$ such that $2 < x_n < 3$ for all $n \in \mathbb{N}$ and $x_n \to 2$. Then

$$\lim_{n\to\infty} \frac{x_n + \lfloor x_n \rfloor}{x_n - \lfloor x_n \rfloor} = \lim_{n\to\infty} \frac{x_n + 2}{x_n - 2},$$

which does not exist. ∎

Some applications of Theorem 4.1:

Theorem 4.2 (**Algebraic properties of limit**). *Let $f, g : E \subset X \to \mathbb{R}$ and a, b be constants. Suppose*

$$\lim_{x\to x_0} f(x) = L, \quad \lim_{x\to x_0} g(x) = M.$$

Then

1. $\displaystyle\lim_{x\to x_0} (af + bg)(x) = aL + bM$
2. $\displaystyle\lim_{x\to x_0} (fg)(x) = LM$
3. *If $M \neq 0$ then* $\displaystyle\lim_{x\to x_0} (f/g)(x) = L/M$.

Proof. Let $\{x_n\}$ be a sequence in $E \setminus \{x_0\}$ such that $x_n \to x_0$. To establish 1–3, we have to show that

$$af(x_n) + bg(x_n) \to aL + bM, \quad f(x_n)g(x_n) \to LM, \quad (f/g)(x_n) \to L/M. \tag{1}$$

Since $f(x) \to L, g(x) \to M$ as $x \to x_0$, we have by Theorem 4.1 that

$$f(x_n) \to L, \quad g(x_n) \to M.$$

So the desired identities above are true by Theorem 3.1. □

Lemma 4.3. *Let $f, g : E \subset X \to \mathbb{R}$. Suppose $f(x) \leqslant g(x)$ for all $x \in E \setminus \{x_0\}$ and $f(x), g(x)$ converge as $x \to x_0$. Then*

$$\lim_{x\to x_0} f(x) \leqslant \lim_{x\to x_0} g(x).$$

Theorem 4.3 (Squeeze theorem). *Let $f, g, h : E \subset X \to \mathbb{R}$. Suppose there is $r > 0$ such that*

$$g(x) \leqslant f(x) \leqslant h(x) \quad \text{for all } x \in E \setminus \{x_0\},\ d(x, x_0) < r$$

and that $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$. Then $\lim_{x\to x_0} f(x) = L$.

Proof. Left as an exercise. □

4.2 Infinite limits and limits at infinity

Now we consider *real-valued* functions defined on a subset E of a metric space X, or defined on an interval containing (a, ∞) or $(-\infty, -a)$ $(a > 0)$.

Definition 4.3 (Infinite limits). Let $f : E \subset X \to \mathbb{R}$.

1. We say that $\lim_{x\to x_0} f(x) = \infty$ if

$$\forall R > 0, \exists \delta > 0 \text{ such that if } x \in E \setminus \{x_0\},\ d(x, x_0) < \delta, \text{ then } f(x) > R.$$

2. We say that $\lim_{x\to x_0} f(x) = -\infty$ if

$$\forall R > 0, \exists \delta > 0 \text{ such that if } x \in E \setminus \{x_0\},\ d(x, x_0) < \delta, \text{ then } f(x) < -R.$$

Example 4.10. Show that $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

Solution. Let $R > 0$. Choose $\delta = 1/\sqrt{R}$. If $x \in \mathbb{R} \setminus \{1\}$ and $|x - 1| < \delta$ then

$$\frac{1}{(x-1)^2} > \delta^2 = R.$$

Thus $1/(x-1)^2 \to \infty$ as $x \to 1$. ■

Lemma 4.4. *Let $f : E \subset X \to \mathbb{R}$.*

1. $\lim_{x\to x_0} f(x) = \infty \Leftrightarrow \lim_{x\to x_0} -f(x) = -\infty$.
2. $\lim_{x\to x_0} f(x) = \infty$ *if and only if* $\exists\, r > 0$ *s.t.* $f(x) > 0$ *for* $x \in (E \cap B_r(x_0)) \setminus \{x_0\}$ *and* $\lim_{x\to x_0} \frac{1}{f(x)} = 0$.
3. $\lim_{x\to x_0} f(x) = -\infty$ *if and only if* $\exists\, r > 0$ *s.t.* $f(x) < 0$ *for* $x \in (E \cap B_r(x_0)) \setminus \{x_0\}$ *and* $\lim_{x\to x_0} \frac{1}{f(x)} = 0$.

Theorem 4.4 (**Sequential characterization of infinite limits**).

1. $\lim\limits_{x\to x_0} f(x) = \infty \Leftrightarrow f(x_n) \to \infty$, *for any* $\{x_n\}$ *in* $E \setminus \{x_0\}$, $x_n \to x_0$.

2. $\lim\limits_{x\to x_0} f(x) = -\infty \Leftrightarrow f(x_n) \to -\infty$, *for any* $\{x_n\}$ *in* $E \setminus \{x_0\}$, $x_n \to x_0$.

Question. Prove the above theorem. "$\Rightarrow$" follows from the definition. "$\Leftarrow$" uses the contrapositive argument. (See the proof of Theorem 4.1.)

Definition 4.4 (**Limit at infinity**). Let $f : E \subset \mathbb{R} \to \mathbb{R}$ and $a > 0$.

1. Suppose $(a, \infty) \subset E$. We write $\lim\limits_{x\to\infty} f(x) = L$ if $\forall\, \varepsilon > 0$, $\exists\, R > a$ such that if $x > R$ then $|f(x) - L| < \varepsilon$.

2. Suppose $(-\infty, -a) \subset E$. We write $\lim\limits_{x\to-\infty} f(x) = L$ if $\forall\, \varepsilon > 0$, $\exists\, R > a$ such that if $x < -R$ then $|f(x) - L| < \varepsilon$.

The definitions for

$$\lim_{x\to\infty} f(x) = \infty \quad \text{or} -\infty$$

and

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{or} -\infty$$

are left as an exercise.

Example 4.11. Show that $\lim\limits_{x\to 0^+} e^{1/x} = \infty$ and $\lim\limits_{x\to 0^-} e^{1/x} = 0$.

Solution. It is easy to observe that the two equalities are equivalent. We prove the first one.

Let $R > 0$. We can assume $R > 1$. Choose $\delta = 1/\ln R$. Observe that $\delta > 0$. If $0 < x < \delta$ then

$$e^{1/x} > e^{1/\delta} = R.$$

So $e^{1/x} \to \infty$ as $x \to 0^+$. ∎

Example 4.12. Show that $\lim\limits_{x\to\infty} (x + x(\sin x)/2) = \infty$.

Solution. Let $R > 0$. Choose $r = 2R$. If $x > r$ then

$$x + x(\sin x)/2 \geqslant x - x/2 = x/2 > R.$$

So $x + x \sin x/2 \to \infty$ as $x \to \infty$. ∎

4.3 Limsup and liminf of functions

Let $f : E \subset X \to \mathbb{R}$. By the sequential characterization of limits, we introduce

$$\mathcal{E} = \{\alpha \in \mathbb{R} : f(x_n) \to \alpha, \exists \{x_n\} \text{ in } E \setminus \{x_0\}, x_n \to x_0\}.$$

Now we can extend the notion of limsup and liminf to functions.

Definition 4.5 (**Limsup and liminf**). Suppose $f : E \subset X \to \mathbb{R}$ is a bounded function. We define

$$\limsup_{x \to x_0} f(x) = \sup \mathcal{E}, \qquad \liminf_{x \to x_0} f(x) = \inf \mathcal{E}.$$

Remark. In the case that f is unbounded (above or below or both)

1. if $f(x_n) \to \infty$, $\exists \{x_n\}$ in $E \setminus \{x_0\}$, $x_n \to x_0$ i.e. $\sup \mathcal{E} = \infty$, we put
$$\limsup_{x \to x_0} f(x) = \infty;$$
2. if $f(x_n) \to -\infty$, $\exists \{x_n\}$ in $E \setminus \{x_0\}$, $x_n \to x_0$, i.e. $\inf \mathcal{E} = -\infty$, we put
$$\liminf_{x \to x_0} f(x) = -\infty.$$

Unlike a limit which may not exist, the limsup and liminf always exist for any given real value function and any limit point x_0.

Theorem 4.5 (**Sequential characterization of limsup and liminf**). *Suppose $f : E \subset X \to \mathbb{R}$ is a bounded function.*

1. $\limsup_{x \to x_0} f(x) = L \Leftrightarrow \limsup_{n \to \infty} f(x_n) \leqslant L$ *for any* $\{x_n\}$ *in* $E \backslash \{x_0\}$ *with* $x_n \to x_0$, *and the equality is attained.*
2. $\liminf_{x \to x_0} f(x) = K \Leftrightarrow \liminf_{n \to \infty} f(x_n) \geqslant K$ *for any* $\{x_n\}$ *in* $E \backslash \{x_0\}$ *with* $x_n \to x_0$, *and the equality is attained.*

Proof. We prove the first statement.

($\Rightarrow$) Let $\varepsilon > 0$. Let $\limsup_{x \to x_0} f(x) = L$. It is clearly that $\limsup_{n \to \infty} f(x_n) \leqslant L$ for any $\{x_n\}$ in $E \setminus \{x_0\}$ with $x_n \to x_0$.

We show that the equality is attained. For each $n \in \mathbb{N}$, there is $\alpha \in \mathcal{E} \cap (L - 1/(2n), L]$ and $x_n \in E \setminus \{x_0\}$ such that $d(x_n, x_0) < 1/n$ and $|f(x_n) - \alpha| < 1/(2n)$. By the triangle inequality, we have

$$|f(x_n) - L| \leqslant |f(x_n) - \alpha| + |\alpha - L| < \frac{1}{n}.$$

We have $\{x_n\}$ as a desired sequence.

($\Leftarrow$) Obvious. □

Example 4.13. Let

$$f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1, \\ 10 & \text{if } x = 1, \\ 2x + 3 & \text{if } x > 1. \end{cases}$$

Find $\limsup_{x \to 1} f(x)$ and $\liminf_{x \to 1} f(x)$.

Solution. It is directely to show that $1, 5$ are the only values α that can be $\lim_{n \to \infty} f(x_n)$, where $x_n \neq 1$ and $x_n \to 1$. Thus $\limsup_{x \to 1} f(x) = 5$ and $\liminf_{x \to 1} f(x) = 0$. ■

Theorem 4.6. *Suppose $f : E \subset X \to \mathbb{R}$ is a bounded function. Then*

$$\limsup_{x \to x_0} f(x) = \limsup_{r \to 0} \{f(x) : x \in (E \cap B_r(x_0)) \setminus \{x_0\}\}$$
$$\liminf_{x \to x_0} f(x) = \liminf_{r \to 0} \{f(x) : x \in (E \cap B_r(x_0)) \setminus \{x_0\}\}$$

Proof. We prove the first equality, the other is left as an exercise.

We apply Theorem 4.5. Let

$$L(r) = \sup\{f(x) : x \in (E \cap B_r(x_0)) \setminus \{x_0\}\}, \quad L = \lim_{r \to 0} L(r).$$

Let $\{x_n\}$ be a sequence in $E \setminus \{x_0\}$ such that $x_n \to x_0$. Then

$$\limsup_{n \to \infty} f(x_n) = \lim_{n \to \infty} \sup\{f(x_k) : k \geqslant n\} \leqslant L.$$

We show that there is $\{x_n\}$ in $E \setminus \{x_0\}$ such that $x_n \to x_0$ and $f(x_n) \to L$. For each n, choose $0 < r_n < 1/(2n)$ such that $|L(r_n) - L| < 1/(2n)$. Then choose $x_n \in (E \cap B_{r_n}(x_0)) \setminus \{x_0\}$ such that $|f(x_n) - L(r_n)| < 1/(2n)$. It follows that

$$|x_n - x_0| < 1/(2n), \quad |f(x_n) - L| < 1/n.$$

So $x_n \to x_0$ and $f(x_n) \to L$. □

Corollary 4.1. *Let $f, g : E \subset X \to \mathbb{R}$ be bounded functions.*

1. $\limsup_{x \to x_0} -f(x) = -\liminf_{x \to x_0} f(x)$.
2. $\limsup_{x \to x_0}[f(x) + g(x)] \leqslant \limsup_{x \to x_0} f(x) + \limsup_{x \to x_0} g(x)$.
3. $\liminf_{x \to x_0}[f(x) + g(x)] \geqslant \liminf_{x \to x_0} f(x) + \liminf_{x \to x_0} g(x)$.
4. *If $c \geqslant 0$ then* $\limsup_{x \to x_0} cf(x) = c \limsup_{x \to x_0} f(x)$, $\liminf_{x \to x_0} cf(x) = c \liminf_{x \to x_0} f(x)$.

5. *If* $f, g \geqslant 0$ *then* $\limsup_{x \to x_0} f(x)g(x) \leqslant \limsup_{x \to x_0} f(x) \limsup_{x \to x_0} g(x)$

6. *If* $f, g \geqslant 0$ *then* $\liminf_{x \to x_0} f(x)g(x) \geqslant \liminf_{x \to x_0} f(x) \liminf_{x \to x_0} g(x)$.

7. *If* $f \leqslant g$ *then* $\limsup_{x \to x_0} f(x) \leqslant \limsup_{x \to x_0} g(x)$, $\liminf_{x \to x_0} f(x) \leqslant \liminf_{x \to x_0} g(x)$.

Proof. Exercise. □

Exercise 4.1.

1. *Let* $f : E \subset X \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, *and* $x_0 \in X$ *be a limit point of* E. *Assume* $\lim_{x \to x_0} f(x) = L$ *and* $\lim_{t \to L} g(t) = M$. *Prove that* $\lim_{x \to x_0} g(f(x)) = M$.

2. *Let* $p(x)$ *be a polynomial and* $x_0 \in \mathbb{R}$. *Prove that* $\lim_{x \to x_0} p(x) = p(x_0)$.

3. *Prove Theorem 4.3.*

4. *Prove Corollary 4.1.*

Mathematical Analysis: Lecture 12

Sujin Khomrutai, Ph.D.

Summary. *In this lecture we study*

1. *the continuity of functions on a metric space,*
2. *sequential characterization of continuity,*
3. *topological characterization of continuity,*
4. *uniform continuity.*

4.4 Continuity of functions

Throughout this note, $(X, d), (Y, \rho)$ are metric spaces. In the previous lecture, we consider functions defined on a subset E of X. But E becomes a metric space with the induced metric from X. In addition, the complement $X \setminus E$ has no effect on f, so from now on it is no loss of generality to study functions defined on X.

Definition 4.6 (Continuity of functions). Let $f : X \to Y$. We say that f is **continuous** at $x_0 \in X$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $d(x, x_0) < \delta$ then

$$\rho(f(x), f(x_0)) < \varepsilon.$$

If f is not continuous at x_0, we say that f is **discontinuous** at x_0.

If f is continuous at every point in E, it is called **continuous**.

Question. Show that f is continuous at every isolated point $x_0 \in X$.

Lemma 4.5. *Let $f : X \to Y$ and $x_0 \in X$ a limit point. We have f is continuous at x_0 if and only if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Proof. ($\Rightarrow$) Suppose f is continuous at x_0. Let $\varepsilon > 0$. Choose $\delta > 0$ such that if $d(x, x_0) < \delta$ then $\rho(f(x), f(x_0)) < \varepsilon$. This implies $\lim_{x\to x_0} f(x) = f(x_0)$.

($\Leftarrow$) Suppose $\lim_{x\to x_0} f(x) = f(x_0)$. For $\varepsilon > 0$, we choose $\delta > 0$ such that for any $0 < d(x, x_0) < \delta$, we get $\rho(f(x), f(x_0) < \varepsilon$. So f is continuous at x_0. □

Example 4.14. Let $f : (-3, 1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x+2 & \text{if } -3 < x < -2, \\ -x-2 & \text{if } -2 \leqslant x < 0, \\ x+2 & \text{if } 0 \leqslant x < 1. \end{cases}$$

Show that f is discontinuous at 0 and continuous on $(-3, 1) \setminus \{0\}$.

Solution. Let $E_1 = (-3, 1) \setminus \{0\}$. If $x_0 \in E_1$ and $x_0 \neq -2$, then

$$\lim_{x\to x_0} f(x) = f(x_0).$$

So f is continuous at x_0. At -2, we have $\lim_{x\to 2^-} f(x) = 0 = \lim_{x\to 2^+} f(x) = f(-2)$, so f is continuous at -2. Thus f is continuous on E_1.

Consider at 0. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x-2) = -2$ but $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+2) = 2$, so $\lim_{x\to 0} f(x)$ does not exist, which implies f is discontinuous at 0. ∎

From now on, we assume X has no isolated point. So

every $x_0 \in X$ is a limit point.

Theorem 4.7 (**Sequential characterization of continuity**)**.** *$f : X \to Y$ is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$ in X.*

Proof. By Lemma 4.5 we have f is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$. Now, by Theorem 4.1, we have $\lim_{x\to x_0} f(x) = f(x_0)$ if and only if $f(x_n) \to f(x_0)$ for any $x_n \to x_0$. So the desired statement is true. □

Example 4.15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at 0 and is discontinuous everywhere else.

Solution. Observe that

$$|f(x)| \leqslant |x| \quad \forall x \in \mathbb{R}.$$

For any $x_n \to 0$, $|f(x_n)| \leqslant |x_n| \to 0$ so $f(x_n) \to 0 = f(0)$. So f is continuous at 0.

If $x_0 \in \mathbb{Q}$ and $x_0 \neq 0$ we take a sequence $\{x_n\}$ in $\mathbb{R} \setminus \mathbb{Q}$ with $x_n \to x_0$. Then $f(x_n) = 0 \not\to x_0$ and $x_0 = f(x_0)$. So f is discontinuous at x_0.

If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ we take $\{x_n\}$ in $\mathbb{Q}$ with $x_n \to x_0$. Then $f(x_n) = x_n \to x_0 \neq f(x_0)$. So f is discontinuous at x_0. ∎

Theorem 4.8 (**Algebraic properties of continuity**). *Suppose $f, g : X \to \mathbb{R}$ are continuous at $x_0 \in X$. Then so are*

$$af + bg, \quad fg, \quad f/g \quad (g(x_0) \neq 0).$$

In particular, if $f, g : X \to \mathbb{R}$ are continuous, then so are $af+bg, fg, f/g(g(x) \neq 0)$

Proof. For $x_n \to x_0$ in X, we get

$$(af + bg)(x_n) \to (af + bg)(x_0), \quad (fg)(x_n) \to (fg)(x_0), \quad (f/g)(x_n) \to (f/g)(x_0),$$

by the algebraic properties of limits of sequences. So the desired assertion is true. □

Example 4.16. Show that $f(x) = c$, $f(x) = x$, and a polynomial $p(x)$ are continuous functions. A rational function $q(x)/r(x)$ is continuous at every $r(x) \neq 0$.

Solution. f, g are continuous by Theorem 4.7. The continuity of $p(x).q(x)/r(x)$ follows from the preceding theorem. ■

Example 4.17. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x, \quad g(x, y) = y.$$

They are called the **projections** of $\mathbb{R}^2$. Show that f, g are continuous.

Solution. Let $(x_n, y_n) \to (x_0, y_0)$ in $\mathbb{R}^2$. This means

$$\|(x_n, y_n) - (x_0, y_0)\| = \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} \to 0 \quad \text{as } n \to \infty.$$

Since $|x_n - x_0| \leqslant \|(x_n, y_n) - (x_0, y_0)\|$, we get $x_n \to x_0$. So $f(x_n, y_n) \to f(x_0, y_0)$, hence f is continuous. Similarly, g is continuous. ■

Remark. The preceding example can be generalized to that for each $j \in \{1, 2, \ldots, k\}$, the function $p_j : \mathbb{R}^k \to \mathbb{R}$ defined by

$$p_j(x_1, \ldots, x_k) = x_j$$

is continuous. The functions p_j are called the **j'th projection** of $\mathbb{R}^k$.

Theorem 4.9 (**Composition of continuous functions**). *Let $f : X \to Y$ and $g : Y \to Z$. Suppose f is continuous at $x_0 \in X$ and g is continuous at $f(x_0)$. Then the composite function $h = g \circ f : X \to Z$ defined by*

$$h(x) = g(f(x))$$

is continuous at x_0. In particular, if f, g are continuous functions, then so is $g \circ f$.

Proof. For $x_n \to x_0$, by Theorem 4.7, we get $f(x_n) \to f(x_0)$ and hence $g(f(x_n)) \to g(f(x_0))$. This means $h(x_n) \to h(x_0)$ so h is continuous at x_0. The second statement follows immediately. □

Example 4.18. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ and

$$F(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y)),$$

where $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$. Prove that F is continuous $\Leftrightarrow$ f_1, f_2, f_3 are continuous.

Solution. ($\Rightarrow$) Let $p_1, p_2, p_3 : \mathbb{R}^3 \to \mathbb{R}$ be the projections of $\mathbb{R}^3$. We have

$$f_j = p_j \circ F.$$

Since F and p_j are continuous functions, we have by Theorem 4.9 that f_1, f_2, f_3 are continuous.

($\Leftarrow$) Let $(x_n, y_n) \to (x_0, y_0)$. We have

$$\|F(x_n, y_n) - F(x_0, y_0)\| = \sqrt{\sum_{j=1}^{3}(f_j(x_n, y_n) - f_j(x_0, y_0))^2}.$$

Since f_j are continuous, we have $f_j(x_n, y_n) \to f_j(x_0, y_0)$ for all $j = 1, 2, 3$. Hence $F(x_n, y_n) \to F(x_0, y_0)$. So F is continuous. ■

Remark. Generally, a function $F : \mathbb{R}^k \to \mathbb{R}^m$, $F = (f_1, \ldots, f_m)$, where $f_j : \mathbb{R}^k \to \mathbb{R}$, is continuous if and only if $f_1, \ldots, f_m$ are continuous functions.

The continuity of $f : X \to Y$ at x_0 can be formulated in term of open balls: f is continuous at x_0 if and only if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)). \tag{2}$$

Theorem 4.10 (**Topological characterization of continuity**). *Let $f : X \to Y$. The following statements are equivalent.*

1. *f is continuous.*
2. *For every open set $U \subset Y$, $f^{-1}(U)$ is an open set in X.*
3. *For every closed set $G \subset Y$, $f^{-1}(G)$ is a closed set in X.*

Proof. 2, 3 are equivalent because

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$$

1 $\Rightarrow$ 2. Suppose f is continuous. Let $U \subset Y$ be an open set and $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since U is an open set, $\exists\, \varepsilon > 0$ such that

$$B_\varepsilon(f(x_0)) \subset U.$$

By Eqn. (2), there is $\delta > 0$ such that $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \subset U$, this implies

$$B_\delta(x_0) \subset f^{-1}(U).$$

So $f^{-1}(U)$ is an open set.

$2 \Rightarrow 1$. Let $x_0 \in X$. We have to prove Eqn. (2). The ball $B_\varepsilon(f(x_0))$ is open in Y, so by assumption 2, we get $V = f^{-1}(B_\varepsilon(f(x_0)))$ is an open set in X. Clearly, $x_0 \in V$. So $\exists\ \delta > 0$ such that

$$B_\delta(x_0) \subset V \quad \Rightarrow \quad f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)).$$

So Eqn. (2) is true hence f is continuous at any x_0. □

4.5 Uniform continuity

If $f : X \to Y$ is a continuous function on X, the "modulus of continuity" of f at each x_0 could depend on the point. By the modulus of continuity, we mean the choice of δ for each $\varepsilon > 0$, which can depend not only on ε but also on x_0.

Definition 4.7 (**Uniform continuity**)**.** A function $f : X \to Y$ is said to be **uniformly continuous** if it has the following property: for all $\varepsilon > 0$ there is a constant $\delta > 0$ such that if $d(x, y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

The uniform continuity will play a vital role in our later discussion.

Theorem 4.11 (**Sequential characteization of uniform continuity**)**.** *A function $f : X \to Y$ is uniformly continuous if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$d(x_n, y_n) \to 0$$

then

$$\rho(f(x_n), f(y_n)) \to 0.$$

Proof. ($\Rightarrow$) We prove the contrapositive. Assume there are sequences $\{x_n\}, \{y_n\}$ such that $d(x_n, y_n) \to 0$ but $\rho(f(x_n), f(y_n)) \not\to 0$.

Then there is $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there is $\rho(f(x_n), f(y_n)) \geqslant \varepsilon$ for some $n \geqslant N$. Thus we get subsequences $\{x_{n_k}\}$, $\{y_{n_k}\}$ such that $\rho(f(x_{n_k}), f(y_{n_k})) \geqslant \varepsilon$ for all k.

Now $d(x_{n_k}, y_{n_k}) \to 0$ but $\rho(f(x_{n_k}), f(y_{n_k})) \not\to 0$, so f is not uniformly continuous.

($\Leftarrow$) Homework. □

Example 4.19. Show that $f(x) = \dfrac{1}{1+x^2}$ is uniformly continuous on $[0, 1]$.

Solution. Consider

$$|f(x_n) - f(y_n)| = |x_n + y_n|(1 + x_n^2)(1 + y_n^2)|x_n - y_n| \leqslant 2|x_n - y_n|.$$

So if $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$. So f is uniformly continuous. ■

Example 4.20. Show that the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous.

Solution. Choose sequences $x_n = 1/n$ and $y_n = 1/(2n)$. Then $|x_n - y_n| = 1/(2n) \to 0$ as $n \to \infty$ and

$$|f(x_n) - f(y_n)| = n \geqslant 1.$$

So $|f(x_n) - f(y_n)| \not\to 0$, hence f is not uniformly continuous. ■

Exercise 4.2.

1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f = g$.*
2. *Let $f : X \to \mathbb{R}$ be a continuous function.*
   (a) *The **zero set** of f is $Z(f) = \{x \in X : f(x) = 0\}$. Prove that $Z(f)$ is a closed set in X.*
   (b) *Prove that $\{x \in X : f(x) > 0\}$ is an open set.*
   (c) *Prove that $\{x \in X : f(x) \leqslant 1\}$ is a closed set.*
3. *The metric d on a metric space is a function $d : X \times X \to \mathbb{R}$. Prove that d is a uniformly continuous function on $X \times X$.*
4. *Let $f : X \to Y$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, prove that $\{f(x_n)\}$ is a Cauchy sequence in Y.*
5. *Prove that if $f, g : X \to \mathbb{R}$ are uniformly continuous, then so are $af + bg, fg, f/g (g(x) \neq 0)$. Also, show that if $f : X \to Y, g : Y \to Z$ are uniformly continuous then so is $g \circ f$.*

■

Mathematical Analysis: Lecture 13

Sujin Khomrutai, Ph.D.

Summary. *In this lecture we study*

1. *Continuity and compactness.*
2. *The extreme value theorem.*
3. *Continuity and connectedness.*

4.6 Continuity and Compactness

Let $(X, d), (Y, \rho)$ be metric spaces.

Recall the set identities:

$$f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$$

and

$$U \subset f^{-1}(f(U)), \quad f(f^{-1}(U)) \subset U.$$

Theorem 4.12 (**Continuity and compactness**). *Let $f : X \to Y$ be a continuous function. Suppose K is a compact subset of X. Then the image $f(K)$ is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$. Since f is continuous, each $f^{-1}(U_\alpha)$ is an open subset of X. Since

$$K \subset \bigcup_{\alpha \in A} f^{-1}(U_\alpha),$$

we find that $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X. Now K is compact, so there is a finite sub-cover, says $f^{-1}(U_1), \dots, f^{-1}(U_N)$. Since $f(f^{-1}(U_j)) \subset U_j$, we get that

$$f(K) \subset f(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_N)) = f(f^{-1}(U_1 \cup \cdots \cup U_N)) \subset U_1 \cup \cdots \cup U_N,$$

we get that $f(K)$ is covered by the finite subcollection $\{U_1, \dots, U_N\}$. So $f(K)$ is compact. $\square$

Theorem 4.13 (Extreme value theorem). *Suppose $f : X \to \mathbb{R}$ is a continuous function and K is a compact subset of X. Define*

$$M = \sup_{x\in K} f(x), \quad m = \inf_{x\in K} f(x).$$

Then m, M are real numbers and there are points $x_1, x_2 \in K$ such that $f(x_1) = m$ and $f(x_2) = M$.

Proof. By Theorem 4.9, $f(K)$ is a compact subset of $\mathbb{R}$ hence it is a closed and bounded subset of $\mathbb{R}$. It follows that M, m are real numbers and they attain at some points in K. □

Example 4.21. Suppose $\{x_n\}$ is a bounded sequence in $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, prove that $\{f(x_n)\}$ is also bounded.

Solution. We can assume $L \leqslant x_n \leqslant M$ for all $n \in \mathbb{N}$, for some constants L, M. Since f is continuous and $[L.M]$ is compact, it follows that $f([L, M])$ is compact. Clearly,

$$f(x_n) \in f([L, M]) \quad \text{for all } n \in \mathbb{N}.$$

Since $f([L, M])$ is a compact subset of $\mathbb{R}$, it is bounded. Thus $\{f(x_n)\}$ is a bounded sequence. ■

Example 4.22. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Suppose $\lim_{x\to\infty} f(x) = 0$. Prove that f is bounded.

Solution. That $\lim_{x\to\infty} f(x) = 0$ implies $\exists\, R > 0$ such that $|f(x)| \leqslant 1$ for all $x \geqslant R$. Now the interval $[0, R]$ is compact. Since $|f|$ is continuous on $[0, R]$, it follows by the extreme value theorem that $M = \sup_{x\in[0,R]} |f(x)|$ is a real number. In particular, we have $|f(x)| \leqslant \max\{1, M\}$ for all $x \in [0, \infty)$, so f is bounded. ■

Example 4.23. We denote

$$C([0, 1]) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

For each $f, g \in C([0, 1])$, we define

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Prove that d is a metric so that $C([0, 1])$ is a metric space.

Solution. Since f, g are continuous, it follows by algebraic properties of continuous functions that $|f - g|$ is continuous. Since $[0, 1]$ is compact, it follows by the extreme value theorem that

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

is a real number.

Clearly, $d(f,g) \geqslant 0$ and $d(f,g) = d(g,f)$. If $d(f,g) = 0$ then $|f(x) - g(x)| \leqslant 0$ for all $x \in K$, hence $f = g$.

For the triangle inequality, we have for each $x \in [0,1]$ that

$$|f(x)-h(x)| \leqslant |f(x)-g(x)|+|g(x)-h(x)| \leqslant \sup_{x\in[0,1]} |f(x)-g(x)|+ \sup_{x\in[0,1]} |g(x)-h(x)|,$$

hence $d(f,h) \leqslant d(f,g) + d(g,h)$ for all $f,g,h \in C([0,1])$. ∎

Remark. *In the preceding example, the closed interval* $[0,1]$ *can be replaced by any compact subset* K *of a metric space.*

Theorem 4.14. *If* $f : [a,b] \to \mathbb{R}$ *is a continuous function, then* f *is uniformly continuous.*

Proof. Let $\varepsilon > 0$. By the continuity of f, for each $x_0 \in [a,b]$, there is $\delta_{x_0} > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{for all } x,y \in I_{x_0} \cap [a,b], \text{ where } I_{x_0} = (x_0 - \delta_{x_0}, x_0 + \delta_{x_0}).$$

Let $I'_{x_0} = (x_0 - \delta_{x_0}/2, x_0 + \delta_{x_0}/2)$. Since $[a,b]$ is compact and $\{I'_{x_0} : x_0 \in [a,b]\}$ is an open cover of $[a,b]$, it follows that there is a finite sub-cover $I'_1, \dots, I'_N$ of lengths $\delta_1/2, \dots, \delta_N/2$ and centers $x_1, \dots, x_N$, respectively.

Let $\delta = \min\{\delta_1/2, \dots, \delta_N/2\} > 0$. Consider $x,y \in [a,b]$ and $|x-y| < \delta$. Then $x \in I'_k$ for some k. So $|x_k - x| < \delta_k/2$. Since $|x_k - y| \leqslant |x_k - x| + |x - y| < \delta_k/2 + \delta \leqslant \delta_k$, it follows that $x, y \in I_k$, so

$$|f(x) - f(y)| < \varepsilon.$$

Thus f is uniformly continuous. □

Remark. *The preceding theorem can be extended any compact metric space. That is, if* $f : X \to Y$ *is continuous and* X *is compact, then* f *is uniformly continuous.*

Example 4.24. Prove that there is a constant $a > 0$ such that if $x, y \in \mathbb{R}$, $|x - y| < a$ then

$$|\sin x - \sin y| < \frac{1}{100}.$$

Solution. We can assume $0 < a < 1$. By trigonometric identity $\sin(x + 2n\pi) = \sin x$, we consider two cases: (i) $x, y \in [0, 2\pi]$ or (ii) $x, y \in [-\pi, \pi]$. In the first case, we apply the theorem above to find $a_1 > 0$ such that $|\sin x - \sin y| < 1/100$. In the second case, we apply the same theorem to get $a_2 > 0$ such that $|\sin x - \sin y| < 1/100$. Let $a = \min\{a_1, a_2\}$. ∎

Example 4.25. If $f : [0, \infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = 0$, prove that f is uniformly continuous.

Solution. Let $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = 0$, $\exists\ R > 0$ such that $|f(x)| < \varepsilon/2$ for all $x > R$. On $[0, R+1]$, f is uniformly continuous by the preceding theorem, so $\exists\ \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0, R+1]$ and $|x - y| < \delta$. We take $\delta < 1$.

Now consider aribtrary $x, y \in [0, \infty)$ with $|x - y| < \delta$.

Case 1. $x, y \in [0, R + 1]$, we are done.

Case 2. $x, y > R$, then $|f(x) - f(y)| \leqslant |f(x)| + |f(y)| < \varepsilon$.

Case 3. $x \leqslant R$ and $y > R + 1$. But this case never happen because, we would get $|x - y| = y - x > 1 > \delta$. ■

4.7 Continuity and connectedness

Theorem 4.15 (Continuity and connectedness). *Let $f : X \to Y$ be a continuous function. If $E \subset X$ is connected, then $f(E)$ is also connected.*

Proof. We use Theorem 2.5. Assume on the contrary that $f(E)$ is disconnected. Then there is a subset A of $f(E)$ such that $\emptyset \subsetneq A \subsetneq f(E)$ and A is both relatively open and relatively closed in $f(E)$. Let $A' = f^{-1}(A) \cap E$. Then $f(A') \subset A \cap f(E) = A$ and clearly it is non-empty, so $\emptyset \subsetneq A' \subsetneq E$.

Since A is open in $f(E)$, $\exists$ an open set U in Y such that $A = U \cap f(E)$. Since A is closed in $f(E)$, $\exists$ a closed set G in Y such that $A = G \cap f(E)$. Now f is continuous, so $f^{-1}(U)$ and $f^{-1}(G)$ are open and closed in X, respectively. It follows that

$$A' = f^{-1}(A) \cap E = f^{-1}(U) \cap E = f^{-1}(G) \cap E,$$

are respectively open and closed in E. This contradicts the connectedness of E. □

Theorem 4.16 (Intermediate value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If c is a number between $f(a)$ and $f(b)$, then there is an $x \in [a, b]$ such that $f(x) = c$.*

Proof. Since f is a continuous function and $[a, b]$ is connected, it follows that $f([a, b])$ is a connected subset of $\mathbb{R}$. By Theorem 2.6, then $f([a, b])$ is an interval. If c is a number between $f(a)$ and $f(b)$, by the characterization of intervals, so there is $x \in [a, b]$ such that $c = f(x)$. □

Example 4.26. Let $f; [a, b] \to [a, b]$ be a continuous function. Prove that there is an $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

Solution. If $f(a) = a$ or $f(b) = b$ then we are done. Assume $f(a) \neq a$ and $f(b) \neq b$. Then $f(a) > a$ and $f(b) < b$.

Let $g : [a, b] \to \mathbb{R}$ be defined by

$$g(x) = f(x) - x \quad (x \in [a, b]).$$

Then g is a continuous function. We have $g(a) = f(a) - a > 0$ and $g(b) < 0$. Since $g(a) > 0 > g(b)$, we have by the intermediate value theorem that $\exists\, g(x_0) = 0$ for some $x_0 \in (a, b)$. So $f(x_0) = x_0$. ■

Example 4.27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Prove that f is surjective.

Solution. Let $y \in \mathbb{R}$ be any real number. That $\lim_{x\to\infty} f(x) = \infty$ implies $\exists\, R_1 > 0$ such that $f(x) > y$ for all $x \geqslant R_1$, and that $\lim_{x\to-\infty} f(x) = -\infty$ implies $\exists\, R_2 > 0$ such that $f(x) < y$ for all $x \leqslant -R_2$. Consider the connected set $[-R_2, R_1]$, it follows by the intermediate value theorem that $\exists\, x$ such that $f(x) = y$. ■

Exercise 4.3.

1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x) \geqslant 0$ for all $x \in \mathbb{R}$. Assume $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$. Prove that f has a maximum value on $\mathbb{R}$.*
2. *Let $f : (0, 1] \to \mathbb{R}$ be a continuous function. Suppose $\lim_{x\to 0^+} f(x) = \infty$. Prove that f is bounded below.*
3. *If $f : X \to Y$ is a continuous, bijection, and X is compact, prove that the inverse function f^{-1} is continuous.*
4. *Prove that the metric space $C([0, 1])$ in Example 4.23 is complete.*
5. *Let X be a connected metric space and $f : X \to \mathbb{R}$. Assume that there are two elements $u, v \in X$ such that $f(u)f(v) < 0$. Prove that the equation $f(x) = 0$ has a solution.*
6. *Let $f, g : X \to \mathbb{R}$ be continuous and X connected such that $f(x) \neq g(x)$ for all x. Prove that $f > g$ or $g > f$ on X.*

■

Mathematical Analysis: Lecture 14

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we study*

1. *derivatives of functions*
2. *some rules for differentiation*
3. *the mean value theorem*

5 Differentiation

5.1 The derivative of a real value function

Definition 5.1 (The definition of derivatives). Let $f : [a, b] \to \mathbb{R}$ be a function and $x_0 \in [a, b]$. We define a number $f'(x_0)$ by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists.

f' is a function whose domain is the set of all x_0 where the limit exists. f' is called the **derivative** of f.

If f' is defined at x_0, we say that f is **differentiable at** x_0. If f' is defined at every point in $E \subset [a, b]$, we say that f is **differentiable on** E.

The derivative function can also be written as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{for } x_0 \in (a, b),$$

and

$$f'(a) = \lim_{h \to 0, h > 0} \frac{f(a + h) - f(a)}{h}, \quad f'(b) = \lim_{h \to 0, h < 0} \frac{f(b + h) - f(b)}{h}.$$

Example 5.1. Let $f(x) = C$, a constant where the domain is $\mathbb{R}$. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{C - C}{x - x_0} = \lim_{x \to x_0} 0 = 0.$$

Thus f is differentiable on $\mathbb{R}$ and its derivative is the zero function. ∎

Example 5.2. Let $f(x) = x^n$ where $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Using the definition

$$f'(x_0) = \lim_{h \to 0} \frac{(x_0 + h)^n - x_0^n}{h}$$

and the binomial expansion

$$\begin{aligned}\frac{(x_0 + h)^n - x_0^n}{h} &= \frac{1}{h}\left(nhx_0^{n-1} + \binom{n}{2}h^2 x_0^{n-2} + \cdots + \binom{n}{n}h^n\right) \\ &= nx_0^{n-1} + \binom{n}{2}hx_0^{n-2} + \cdots + \binom{n}{n}h^{n-1}.\end{aligned}$$

So by taking $h \to 0$, we obtain

$$f'(x_0) = nx_0^{n-1}.$$

Thus $f(x) = x^n$ is differentiable on $\mathbb{R}$ and its derivative is $f'(x) = nx^{n-1}$. ∎

Example 5.3. Let f be differentiable at a. Use the limit definition of derivative to compute

$$\lim_{h \to 0} \frac{f(a + h^2) - f(a - h)}{h}.$$

Solution. We express the quotient as

$$\frac{f(a + h^2) - f(a)}{h} - \frac{f(a - h) - f(a)}{h} = \frac{f(a + h^2) - f(a)}{h^2} \cdot h + \frac{f(a - h) - f(a)}{-h},$$

so by taking $h \to 0$, we find that the limit is equal to

$$f'(a) \cdot 0 + f'(a) = f'(a).$$

Alternatively: Use L'Hospital's rule. ∎

Example 5.4. Let f be differentiable at a and $f(x) \geqslant 0$ in a neighborhood of a with $f(a) = 0$. Prove that $f'(a) = 0$.

Solution. Let say $f(x) \geqslant 0$ for all $x \in (a - \delta, a + \delta)$ where $\delta > 0$. Since $f(a) = 0$, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h)}{h}.$$

Taking $0 < h < \delta$, we find that $f'(a) \geqslant 0$, whereas taking $-\delta < h < 0$, we obtain that $f'(a) \leqslant 0$. Thus $f'(a) = 0$. ∎

Theorem 5.1. *If f is differentiable at a point $x_0 \in [a,b]$, then f is continuous at x_0. So, if f is differentiable on $[a,b]$ then f is uniformly continuous on $[a,b]$.*

Proof. Assume f' is defined at x_0. We have

$$\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0) = f'(x_0)\lim_{x\to x_0}(x-x_0) = 0,$$

by the limit theorem. Thus $\lim_{x\to x_0} f(x) = f(x_0)$ which means f is continuous at x_0.

For the second assertion, we employ the fact that every continuous function is uniformly continuous on compact subset (cf. Theorem 4.14). □

Example 5.5. Let $f(x) = x^n$ ($n \in \mathbb{N}$) where $x \in \mathbb{R}$. Prove that f is uniformly continuous on any interval $[a,b]$.

Solution. We know that $f'(x) = nx^{n-1}$. So f is differentiable on $\mathbb{R}$. In particular, f is differentiable on $[a,b]$. Applying the preceding theorem, we conclude that f is uniformly continuous on $[a,b]$. ■

Theorem 5.2. *Let $f,g : [a,b] \to \mathbb{R}$ be functions which are differentiable at a point $x_0 \in [a,b]$. Then $af+bg, fg, f/g$ ($g'(x_0) \neq 0$) are differentiable at x_0 and*

$$\begin{aligned}(af+bg)'(x_0) &= af'(x_0)+bg'(x_0),\\ (fg)'(x_0) &= f'(x_0)g(x_0)+f(x_0)g'(x_0),\\ \left(\frac{f}{g}\right)'(x_0) &= \frac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}\end{aligned}$$

Proof. For the first part, we consider

$$\begin{aligned}\frac{(af+bg)(x)-(af+bg)(x_0)}{x-x_0} &= a\frac{f(x)-f(x_0)}{x-x_0}+b\frac{g(x)-g(x_0)}{x-x_0},\\ \therefore \lim_{x\to x_0}\frac{(af+bg)(x)-(af+bg)(x_0)}{x-x_0} &= a\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+b\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\\ &= af'(x_0)+bg'(x_0).\end{aligned}$$

So $af+bg$ is differentiable at x_0.

For the second identity, we consider

$$\begin{aligned}\frac{(fg)(x)-(fg)(x_0)}{x-x_0} &= \frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0},\\ \lim_{x\to x_0}\frac{(fg)(x)-(fg)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}g(x)+\lim_{x\to x_0}f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\\ &= f'(x_0)g(x_0)+f(x_0)g'(x_0),\end{aligned}$$

So fg is differentiable at x_0. We have used that if g is differentiable x_0, then g is continuous at x_0 to get that $g(x) \to g(x_0)$ as $x \to x_0$.

For the last identity, we consider

$$\begin{aligned}
\frac{(f/g)(x) - (f/g)(x_0)}{x - x_0} &= \frac{1}{g(x)g(x_0)} \frac{f(x)g(x_0) - f(x_0)g(x)}{x - x_0} \\
&= \frac{1}{g(x)g(x_0)} \left\{ \frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right\}, \\
\lim_{x \to x_0} \frac{(f/g)(x) - (f/g)(x_0)}{x - x_0} &= \frac{1}{g(x_0)^2} \left\{ f'(x_0)g(x_0) - f(x_0)g'(x_0) \right\}.
\end{aligned}$$

So (f/g) is differentiable at x_0. □

Example 5.6. Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial. Then

$$\begin{aligned}
p'(x) &= a_n (x^n)' + a_{n-1}(x^{n-1})' + \cdots + (a_2 x^2)' + (a_1 x)' + a_0', \\
&= a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} \cdots + a_2 2x + a_1.
\end{aligned}$$

so p is differentiable on $\mathbb{R}$.

If $r(x) = p(x)/q(x)$ is a rational function, that is p, q are polynomials, then

$$r'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{q(x)^2}$$

for all x such that $q(x) \neq 0$. So $r(x)$ is differentiable on $\mathbb{R} \setminus Z(g)$. ■

Example 5.7. Let $n \in \mathbb{Z}$. If $n \geqslant 0$, we already know that

$$(x^n)' = n x^{n-1} \quad \text{for all } x \in \mathbb{R}.$$

Now for $n < 0$, we can express $x^n = 1/x^{-n}$ where $-n > 0$. So for $x \neq 0$, we have

$$(x^n)' = \frac{n x^{-n-1}}{x^{-2n}} = n x^{n-1} \quad \text{for all } x \neq 0.$$

We conclude that the formula $(x^n)' = n x^{n-1}$ is true when $n < 0$ as well. ■

Theorem 5.3 (The chain rule). *Let $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ with $f([a, b]) \subset [c, d]$. Suppose f is a continuous function, f is differentiable at a point $x_0 \in [a, b]$, and g is differentiable at $f(x_0)$. Then the function*

$$h(x) = g(f(x)) \quad \text{for } a \leqslant x \leqslant b.$$

is differentiable at x_0 and $h'(x_0) = f'(g(x_0)) \cdot g'(x_0)$.

Proof. Let $y_0 = f(x_0)$ and $y = f(x)$. We define the function $\phi : [a, b] \setminus \{x_0\} \to \mathbb{R}$ by

$$\phi(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \quad \therefore\ f(x) - f(x_0) = (x - x_0)\{\phi(x) + f'(x_0)\}.$$

Similarly, we define $\psi : [c, d] \setminus \{y_0\} \to \mathbb{R}$ by

$$\psi(x) = \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) \quad \therefore\ g(y) - g(y_0) = (y - y_0)\{\psi(y) + g'(y_0)\}.$$

Note that $\lim_{x \to x_0} \phi(x) = 0$ and $\lim_{y \to y_0} \psi(y) = 0$.

Consider

$$\begin{aligned} \frac{h(x) - h(x_0)}{x - x_0} &= \frac{g(y) - g(y_0)}{x - x_0} = \frac{y - y_0}{x - x_0}\{\psi(y) + g'(y_0)\} \\ &= \frac{f(x) - f(x_0)}{x - x_0}\{\psi(f(x)) + g'(f(x_0))\} \\ &= \{\phi(x) + f'(x_0)\}\{\psi(f(x)) + g'(f(x_0))\}. \end{aligned}$$

Taking $x \to x_0$, we get $\phi(x) \to 0$ and $\psi(f(x)) \to 0$, hence h is differentiable at x_0 and $h'(x_0) = g'(f(x_0))f'(x_0)$. □

Example 5.8. Let $r = m/n$ be a rational number, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and n does not divide m. Prove that

$$(x^r)' = rx^{r-1} \quad \text{for } x \neq 0.$$

Solution. Let $f(x) = x^m$ and $g(x) = x^{1/n}$. We have

$$f'(x) = mx^{m-1}.$$

Also by the exercise below we have $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ for all $x \neq 0$. Since $x^r = g(f(x))$ we conclude that x^r is differentiable at every $x \neq 0$ and

$$(x^r)' = g'(f(x))f'(x) = \frac{1}{n}(x^m)^{\frac{1}{n}-1}mx^{m-1} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

So $(x^r)' = rx^{r-1}$. ■

Example 5.9. Let f be differentiable at c and let $\{a_n\}, \{b_n\}$ be sequences such that $a_n < c < b_n$ for all n and $a_n, b_n \to c$. Prove that

$$\lim_{n \to \infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(c).$$

Solution. Define for $x \neq c$, $\phi(x) = \dfrac{f(x) - f(c)}{x - c} - f'(c)$. Then

$$\lim_{x \to c} \phi(x) = 0.$$

We have $f(x) - f(c) = (x - c)(\phi(x) + f'(c))$ and

$$f(b_n) - f(a_n) = (b_n - c)(\phi(b_n) + f'(c)) - (a_n - c)(\phi(a_n) + f'(c)),$$
$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = \frac{(b_n - c)\phi(b_n) - (a_n - c)\phi(a_n)}{b_n - a_n} + f'(c),$$

that $\phi(b_n), \phi(a_n) \to 0$ and the sequeeze theorem implies

$$\left|\frac{(b_n - c)\phi(b_n) - (a_n - c)\phi(a_n)}{b_n - a_n}\right| \leqslant |\phi(b_n)| + |\phi(a_n)| \to 0,$$

so the desired identity is true. ■

5.2 The mean value theorem

If $f'(x_0) = 0$ or $f'(x_0)$ does not exist, x_0 is called a **critical point** for f.

Lemma 5.1 (**Local extrema are critical points**). *Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose f has a local maximum (or a local minimum) at $x_0 \in (a, b)$. Then x_0 is a critical point for f.*

Proof. If $f'(x_0)$ does not exist, we are done. Suppose $f'(x_0)$ exists. We assume f has a local minimum at x_0 let says on $(x_0 - \delta, x_0 + \delta)$ where $\delta > 0$. Then $f(x) \geqslant f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. So

$$\frac{f(x) - f(x_0)}{x - x_0} \geqslant 0 \quad \text{for } x_0 < x < x_0 + \delta$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \leqslant 0 \quad \text{for } x_0 - \delta < x < x_0.$$

Since $f'(x_0)$ exists, we have $f'(x_0) \geqslant 0$ from the first inequality and $f'(x_0) \leqslant 0$ from the second one. Thus $f'(x_0) = 0$.

If f has a local maximum, we can apply the previous case to $-f$. □

Question. If $f'(x) = 0$ for all $x \in (a, b)$, is it true that $f(x) =$ a constant?

Theorem 5.4 (**Rolle's theorem**). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose f is differentiable on (a, b) and $f(a) = f(b)$. Then there is $x_0 \in (a, b)$ such that*

$$f'(x_0) = 0.$$

Proof. If $f =$ a constant, the desired conclusion is true. Assume $f(x) \neq f(a)$ for some $x \in (a, b)$.

If $f(x) > f(a)$ we have by the extreme value theorem that f attains an absolute maximum at some $x_0 \in (a, b)$. On the other hand, if $f(x) < f(a)$ we have that f attains an absolute minimum at some $x_0 \in (a, b)$. In either case, we can apply Lemma 5.1 to conclude that $f'(x_0) = 0$ at some $x_0 \in (a, b)$. □

Example 5.10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that f is 1-1.

Solution. Assume there were $x < y$ such that $f(x) = f(y)$. Then by Rolle's theorem, $\exists\ x < z < y$ such that $f'(z) = 0$, which is a contradiction. ∎

Theorem 5.5 (The mean value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose f is differentiable on (a, b). Then there is a point $x_0 \in (a, b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We consider the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x.$$

Then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b) = (bf(a) - af(b))/(b - a)$. So the desired conclusion is true by Rolle's theorem. □

Corollary 5.1. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) \geqslant 0$ for all $x \in (a, b)$, then f is increasing.*
2. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*
3. *If $f'(x) \leqslant 0$ for all $x \in (a, b)$, then f is decreasing.*

In case 1, if $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing; and in case 3, if $f'(x) < 0$ then f is strictly decreasing.

Proof. 1. Consider any $y_1, y_2 \in (a, b)$ with $y_1 < y_2$. By the mean value theorem, $\exists$ $z \in (y_1, y_2)$ such that $f(y_2) - f(y_1) = f'(z)(y_2 - y_1) \geqslant 0$. So $f(y_2) \geqslant f(y_1)$, i.e. f is increasing.

2. As above, we have $f(y_2) - f(y_1) = f'(z)(y_2 - y_1) = 0$. So f is constant.

3. Similarly, we have $f(y_2) - f(y_1) = f'(z)(y_2 - y_1) \leqslant 0$. So f is decreasing. □

The following result will be used in the proof of L'Hospital's rule.

Theorem 5.6 (Cauchy's mean value theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Suppose f, g are differentiable on (a, b). Then there is $x_0 \in (a, b)$ such that*

$$\{f(b) - f(a)\}g'(x_0) = \{g(b) - g(a)\}f'(x_0).$$

Proof. Let
$$h(x) = \{f(b) - f(a)\}g(x) - \{g(b) - g(a)\}f(x).$$
Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = f(b)g(a) - g(b)f(a)$. So the desired conclusion follows from Rolle's theorem. □

Example 5.11. Let $f : \mathbb{R} \to \mathbb{R}$ be such that
$$|f(x) - f(y)| \leqslant (x - y)^2 \quad \text{for all } x, y \in \mathbb{R}.$$
Prove that f is a constant function.

Solution. Consider the ratio
$$\left|\frac{f(x+h) - f(x)}{h}\right| \leqslant |h| \quad \text{for all } h \neq 0.$$
Then we get by the squeeze theorem that
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0.$$
So we conclude that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Employing Corollary 5.1, we conclude that f is a constant. ■

Example 5.12. Prove that there is exactly one solution to the equation $\cos x = \sqrt{x} - 1$ for $x \in (0, \pi/2)$.

Solution. The desired conclusion is the same as there is exactly one solution to the equation $x = (1 + \cos x)^2$.

Let $f(x) = x-(1+\cos x)^2$. Then $f'(x) = 1+2(1+\cos x)\sin x$. Since $\cos x, \sin x \geqslant 0$ for $x \in (0, \pi/2)$, we get $f'(x) > 0$. So f is an increasing function.

Since $f(0) = -1$ and $f(\pi/2) = \pi/2 - 1 > 0$, we have by the intermediate value theorem that there is $x_0 \in (0, \pi/2)$ such that $f(x_0) = 0$. If there were two solutions $x_1 < x_2$, then we would have an $x_3 \in (x_2, x_3)$ such that $f'(x_3) = 0$ which is impossible because $f'(x) > 0$ for all $x \in (0, \pi/2)$. ■

Example 5.13. Prove that
$$|\sin x| \leqslant |x|.$$
Solution. By the mean value theorem, we have
$$|\sin x - \sin 0| = |\cos x_0||x - 0|$$
for some x_0 between 0 and x. So the desired inequality is true by that $|\cos x_0| \leqslant 1$. ■

Example 5.14. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Assume f, g are differentiable on (a, b) and $|f'(x)| \leqslant |g'(x)|$ for all $x \in (a, b)$. If $g'(x) \neq 0$ for all $x \in (a, b)$, prove that
$$|f(x) - f(y)| \leqslant |g(x) - g(y)| \quad \forall\, x, y \in [a, b].$$

Solution. We may assume $x < y$. By the Cauchy's mean value theorem, $\exists\, z \in (x, y)$ such that

$$\{f(y) - f(x)\}g'(z) = \{g(y) - g(x)\}f'(z).$$

This implies

$$|f(y() - f(x)| \leqslant \frac{|f'(z)|}{|g'(z)|}|g(y) - g(x)| \leqslant |g(y) - g(x)|,$$

where we have used that $z \in (a, b)$ so $|f'(z)|/|g'(z)| \leqslant 1$. ■

Example 5.15. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) \to 0$ as $x \to +\infty$. Let $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution. For each $x > 0$, there is $z \in (x, x + 1)$ such that

$$g(x) = f(x + 1) - f(x) = f'(z)(x + 1 - x) = f'(z).$$

Observe that if $x \to +\infty$, then $z > x$ also approaches $+\infty$. Using that $f'(z) \to 0$ as $z \to +\infty$, we have $g(x) \to 0$ as $x \to +\infty$. ■

Exercise 5.1.

1. *Let $f, g : (a, b) \to \mathbb{R}$ be a differentiable function. If $f'(x) = g'(x)$ for all $x \in (a, b)$, prove that there is a constant $c > 0$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

2. *Let $P(x)$ be a polynomial of degree $n \in \mathbb{N}$ and $a \neq 0$. Prove that the equation $e^{ax} = P(x)$ has at most $n + 1$ solutions.*

3. *Assume $f'(x) > 0$ on (a, b). Prove that f is strictly increasing on (a, b) and its inverse f^{-1} is differentiable on (a, b) with*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)} \quad \text{for } x_0 \in (a, b).$$

4. *Prove Bernoulli's inequality: $(1 + x)^r \geqslant 1 + rx$ for all $x \geqslant -1$ and all $r \geqslant 1$.*

5. *If $C_0, C_1, \ldots, C_n$ are constants such that*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n + 1} = 0,$$

prove that the equation $C_0 + C_1 x + \cdots + C_n x^n = 0$ has at least one solution on $(0, 1)$.

■

Mathematical Analysis: Lecture 15

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we study*

1. *L'Hospital's rule*
2. *Higher derivatives*
3. *the Taylor's theorem*

5.3 L'Hospital's Rules

We present one of the most important tools in mathematical analysis. Throughout this note

$$f, g : I \to \mathbb{R} \text{ are differentiable functions}, \quad I \text{ is an interval.}$$

Theorem 5.7 (**L'Hospital's rule 1**). *Let $f, g : I \to \mathbb{R}$ be differentiable functions and $(a, b) \subset I$. Suppose $g'(x) \neq 0$ for all $x \in (a, b)$ and*

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0, \quad \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Proof. Let $a < x < y < b$. First note that $g(x) \neq g(y)$, since otherwise, we could have $g'(\xi) = 0$ for some $\xi \in (x, y)$ by Rolle's theorem.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f'(\xi)/g'(\xi) - L| < \varepsilon$ for all $\xi \in (a, a + \delta)$. Now if $a < x < y < a + \delta$, we have by Cauchy's mean value theorem that $\exists\, \xi \in (x, y) \subset (a, a + \delta)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(\xi)}{g'(\xi)} \quad \Rightarrow \quad \left| \frac{f(y) - f(x)}{g(y) - g(x)} - L \right| < \varepsilon.$$

Fixing $y \in (a, a+\delta)$ and using that $f(x), g(x) \to 0$ as $x \to a$, we get

$$\left| \frac{f(y)}{g(y)} - L \right| < \varepsilon.$$

This implies $\lim_{y \to a^+} f(y)/g(y) = L$. □

Example 5.16. Suppose $f, g : I \to \mathbb{R}$ be differentiable, $(a, b) \subset I$, and

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^-} g(x) = 0, \quad \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L.$$

Then $\lim_{x \to b^-} \dfrac{f(x)}{g(x)} = L$.

Solution. Consider $F(x) = f(-x)$ and $G(x) = g(-x)$ which are defined on $(-b, -a)$. Now F, G are differentiable and

$$\lim_{x \to (-b)^+} F(x) = \lim_{x \to (-b)^+} G(x) = 0, \quad \lim_{x \to (-b)^+} \frac{F'(x)}{G'(x)} = L.$$

So by the L'Hospital's rule 1, we obtain

$$\lim_{x \to (-b)^+} \frac{F(x)}{G(x)} = L \quad \Rightarrow \quad \lim_{x \to b^-} \frac{f(x)}{g(x)} = L,$$

which is the desired conclusion. ■

Example 5.17. Suppose $f, g : I \to \mathbb{R}$ be differentiable, $x_0 \in (a, b) \subset I$, and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0, \quad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L.$$

Then $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = L$.

Solution. We obtain the desired limit identity by applying the previous example and L'Hospital's rule 1. ■

Example 5.18. Calculate the limit

$$\lim_{x \to 0} \frac{x - \tan x}{x}.$$

Solution. Let $f(x) = x - \tan x$ and $g(x) = x$. Check: f, g are differentiable on $\mathbb{R}$, $g'(x) \neq 0$, $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Now calculate

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1 - \sec^2 x}{1} = 0.$$

By the L'Hospital's rule, we get

$$\lim_{x \to 0} \frac{x - \tan x}{x} = 0.$$

■

Example 5.19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose f' is continuous. Prove that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{h} = 2f'(x).$$

Solution. Fix $x \in \mathbb{R}$. We consider functions $F(h) = f(x+h) - f(x-h)$ and $G(h) = h$. Check: F, G are differentiable on $\mathbb{R}$, $G'(h) = 1 \neq 0$, $F(h) \to 0, G(h) \to 0$ as $h \to 0$, and $F'(h)/G'(h) = f'(x+h) + f'(x-h) \to 2f'(x)$ as $h \to 0$. Then applying the L' Hospital's rule

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{h} = \lim_{h\to 0} \frac{f'(x+h) + f'(x-h)}{1} = 2f'(x),$$

as desired. ∎

Example 5.20. Prove the L'Hospital's rule with $a = -\infty$ and $L \in \mathbb{R}$.

Solution. Let $F(x) = f(-1/x)$ and $G(x) = g(-1/x)$. Then F, G are differentiable on $(0, b)$ (some $b > 0$), $G'(x) \neq 0$, $F(x) \to 0, G(x) \to 0$ as $x \to 0^+$, and

$$\lim_{x\to 0^+} \frac{F'(x)}{G'(x)} = \lim_{x\to 0^+} \frac{f'(-1/x)\cdot(1/x^2)}{g'(-1/x)\cdot(1/x^2)} = \lim_{x\to -\infty} \frac{f'(x)}{g'(x)} = L.$$

By L'Hospital's rule 1, we get $f(x)/g(x) \to L$ as $x \to -\infty$. ∎

Example 5.21. Prove the L'Hospital's rule with $a \in \mathbb{R}$ and $L = \infty$.

Solution.. That $f'(x)/g'(x) \to \infty$ as $x \to a^+$ implies $f'(x) \neq 0$ on some $(a, a+\delta)$. Switch the role of f, g. We have $f'(x) \neq 0$ on $(a, a+\delta)$, $f(x) \to 0, g(x) \to 0$ as $x \to a^+$, and

$$\lim_{x\to a^+} \frac{g'(x)}{f'(x)} = 0.$$

By the L'Hospital's rule 1, we conclude that $g(x)/f(x) \to 0$. It can be easily seen that $f(x)/g(x) > 0$ on some $(a, a+\delta)$, hence $f(x)/g(x) \to \infty$ as $x \to a^+$. ∎

Theorem 5.8 (**L' Hospital's rule 2**). *Let $f.g : I \to \mathbb{R}$ be differentiable functions and $(a, b) \subset I$. Suppose $g'(x) \neq 0$ for all $x \in (a, b)$ and*

$$\lim_{x\to a^+} g(x) \in \{\infty, -\infty\}, \quad \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

Proof. Let $a < x < y < b$. By Rolle's theorem, $g(x) \neq g(y)$. Employ Cauchy's mean value theorem to get $\xi \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(\xi)}{g'(\xi)}$$

We rewrite the equation as

$$\frac{f(x)}{g(x)} = \frac{1}{g(x)}\left(f(y) - \frac{f'(\xi)}{g'(\xi)}g(y)\right) + \frac{f'(\xi)}{g'(\xi)}.$$

So

$$\left|\frac{f(x)}{g(x)} - L\right| \leqslant \frac{1}{|g(x)|}\left|f(y) - \frac{f'(\xi)}{g'(\xi)}g(y)\right| + \left|\frac{f'(\xi)}{g'(\xi)} - L\right|.$$

Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that $|f'(\xi)/g'(\xi) - L| < \varepsilon$ for $\xi \in (a, a+\delta_1)$. Fix $y \in (a, a+\delta_1)$. Since $g(x) \to \pm\infty$ as $x \to a$, the term on the right hand side is bounded by 2ε by taking $x \in (a, a+\delta_2)$. This implies $\lim_{x\to a} f(x)/g(x)$ exists and is equal to L. □

Remark. The L'Hospital's rule 2 can be extended to

1. $x \to b^-$, $x \to x_0$
2. $x \to -\infty$, $x \to \infty$
3. $L \in \{-\infty, \infty\}$.

Example 5.22. Let $f : (0,1) \to \mathbb{R}$ be a function. Suppose f is differentiable on $(0,1)$ and $\lim\limits_{x\to 0^+} x^2 f'(x) = L \in \mathbb{R}$. Prove that

$$\lim_{x\to 0^+} xf(x) \quad \text{exists in } \mathbb{R}.$$

Also find the limit in terms of L.

Solution. Applying the L'Hospital's rule to $f(x)/x^{-1}$ (f differentiable, $g(x) = x^{-1}$ satisfies $g'(x) \neq 0$ on $(0,1)$, $g(x) \to \infty$ as $x \to 0^+$, and $f'(x)/g'(x) = -x^2 f'(x) \to -L$ as $x \to 0^+$) we get

$$\lim_{x\to 0^+} xf(x) = \lim_{x\to 0^+} \frac{f(x)}{x^{-1}} = -L.$$

■

Example 5.23. Let $f : [0,\infty) \to \mathbb{R}$ be an n-times differentiable function on $(0,\infty)$, i.e. $f^{(n)}$ exists on $(0,\infty)$. Suppose $\lim\limits_{x\to\infty} f^{(n)}(x) = L \in \mathbb{R}$. Prove that

$$\lim_{x\to\infty} \frac{f(x)}{x^n} \quad \text{exists in } \mathbb{R}$$

and calculate the limit.

Solution. Let $g(x) = x^n$. Applying the L'Hospital's rule to the pair $f^{(n-1)}(x)$ and $g^{(n-1)}(x) = n!x$, we find that

$$\lim_{x\to+\infty} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = \lim_{x\to+\infty} \frac{f^{(n)}(x)}{n!} = L/n! \quad \text{exists in } \mathbb{R}.$$

Then applying the same to $f^{(n-2)}(x)$ and $g^{(n-2)}(x) = n(n-1)\cdots 3x^2$, we find that

$$\lim_{x\to+\infty} \frac{f^{(n-2)}(x)}{g^{(n-2)}(x)} = \lim_{x\to+\infty} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = L/n!.$$

Continuing the process, we obtain that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L/n!$$

This prove the desired result. ■

5.4 Higher Derivatives and Taylor's theorem

Definition 5.2. Let f be a real value function. If f' exists in an interval and it is differentiable, we denote the derivative of f' by f'' and call f'' the **second derivative** of f. Continuing in the same manner, we can define

$$f, \quad f', \quad f'', \quad f''', \quad \dots, \quad f^{(n)}, \dots$$

Example 5.24. Let r_0, r_1, r_2, α be given real numbers. Find a polynomial P of degree at most 2 such that

$$P(\alpha) = r_0, \quad P'(\alpha) = r_1, \quad P''(\alpha) = r_2.$$

Solution. If a_0, a_1, a_2 are real numbers, then

$$Q(x) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2$$

has the property that $Q(\alpha) = a_0$, $Q'(\alpha) = a_1$, $Q''(\alpha) = a_2 2!$. Choosing $a_k = r_k/k!$ $(k = 0, 1, 2)$, then

$$P(x) = r_0 + r_1(x-\alpha) + \frac{r_2}{2!}(x-\alpha)^2$$

satisfies the desired identity. ■

Of course, a polynomial of degree at most n satisfying

$$P(\alpha) = r_0, \quad P'(\alpha) = r_1, \dots, P^{(n)}(\alpha) = r_n$$

for given numbers $r_0, r_1, \dots, r_n$ is

$$P(x) = \sum_{k=0}^{n} \frac{r_k}{k!}(x-\alpha)^k.$$

Definition 5.3. Let $f : [a, b] \to \mathbb{R}$ and $\alpha \in [a, b]$ be such that $f^{(k)}(\alpha)$ exist for $k = 1, \dots, n$. The polynomial

$$P_n(x) = f(\alpha) + \frac{f'(\alpha)}{1!}(x-\alpha) + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$$

which is a polynomial of degree at most n satisfying

$$P_n(\alpha) = f(\alpha), \quad P_n'(\alpha) = f'(\alpha), \quad P_n''(\alpha) = f''(\alpha), \quad \dots, \quad P_n^{(n)}(\alpha) = f^{(n)}(\alpha)$$

is called a **Taylor polynomial** of f at α. By convention, we put

$$P_0(x) = f(\alpha).$$

Example 5.25. Find the Taylor polynomial of degree n for $f(x) = e^x$ at $\alpha = 0$.

Solution. We have

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f^{(n)}(x) = e^x \;\forall\, n \in \mathbb{N}.$$

So $f^{(n)}(0) = e^0 = 1$ and hence

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

■

Example 5.26. Find the Taylor polynomial of degree n for $f(x) = \ln(1+x)$ at $x = 0$.

Solution. We have

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1!}{(1+x)^2}, \quad f^{(3)}(x) = \frac{2!}{(1+x)^3},$$

and

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \quad \forall\, n \in \mathbb{N}.$$

So we have $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ and hence

$$P_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n}x^n.$$

■

We recall, the mean value theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and f' is defined on (a, b), then for any $x_0, \alpha \in [a, b]$ $\exists\, \xi$ between x_0 and α such that $f(x_0) - f(\alpha) = f'(\xi)(x_0 - \alpha)$. The equality can be written as

$$f(x_0) = f(\alpha) + f'(\xi)(x_0 - \alpha).$$

Alternative proof. Choose $M \in \mathbb{R}$ so that

$$f(x_0) = f(\alpha) + M(x_0 - \alpha).$$

Define

$$g(x) = f(x) - (f(\alpha) + M(x - \alpha)), \quad x \in [a, b].$$

Then $g(\alpha) = g(x_0) = 0$, so we get by Rolle's theorem that $\exists\, \xi$ between x_0, α such that

$$g'(\xi) = 0 \quad \Rightarrow \quad M = f'(\xi).$$

The above proof can lead to the following general situation.

Suppose f'' exists on (a, b) and $x_0, \alpha \in [a, b]$.

Claim. There exists ξ between x_0, α such that

$$f(x_0) = f(\alpha) + f'(\alpha)(x_0 - \alpha) + \frac{f''(\xi)}{2!}(x_0 - \alpha)^2.$$

Proof. Let M be a real number so that

$$f(x_0) = f(\alpha) + f'(\alpha)(x_0 - \alpha) + \frac{M}{2!}(x_0 - \alpha)^2.$$

Define

$$g(x) = f(x) - \left(f(\alpha) + f'(\alpha)(x_0 - \alpha) + \frac{M}{2!}(x_0 - \alpha)^2\right), \quad x \in [a, b].$$

Then $g(\alpha) = g(x_0) = 0$, so we get by Rolle's theorem that $\exists\, x_1$ between x_0, α such that

$$g'(x_1) = 0.$$

Now observe that $g'(\alpha) = g'(x_1) = 0$, so we again get by Rolle's theorem that $\exists\, \xi$ between x_1 and α such that

$$g''(\xi) = 0 \quad \Rightarrow \quad f''(\xi) = M.$$

Theorem 5.9 (Taylor's theorem)**.** *Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be a function. Suppose that $f^{(k)}$ exist on $[a, b]$ for $k = 1, \ldots, n - 1$ and $f^{(n)}$ exists for every $x \in (a, b)$. Then for any $x_0, \alpha \in [a, b]$, $\exists\, \xi$ between x_0 and α such that*

$$f(x_0) = f(\alpha) + \frac{f'(\alpha)}{1!}(x_0 - \alpha) + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(x_0 - \alpha)^{n-1} + \frac{f^{(n)}(\xi)}{n!}(x_0 - \alpha)^n.$$

That is, in terms of the Taylor polynomial of f about α, the Taylor's theorem can be stated as

$$f(x_0) = P_{n-1}(x_0) + \frac{f^{(n)}(\xi)}{n!}(x_0 - \alpha)^n.$$

Proof. Let M be a number such that

$$f(x_0) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x_0-\alpha)^k + M(x_0-\alpha)^n = P_{n-1}(x_0) + \frac{M}{n!}(x_0-\alpha)^n.$$

Define the function

$$g(x) = f(x) - \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k + \frac{M}{n!}(x-\alpha)^n\right), \quad x \in [a,b].$$

Observe that $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$ and $g^{(n)}(x) = f^{(n)}(x) - M$.

Since $g(x_0) = 0$ by the choice of M and clearly $g(\alpha) = 0$, we get by Rolle's theorem that $\exists\ x_1$ between x_0 and α such that $g'(x_1) = 0$.

Since $g'(x_1) = 0$ and $g'(\alpha) = 0$, using Rolle's theorem again $\exists\ x_2$ between x_1 and α such that $g''(x_2) = 0$.

Continuing the process, we finally obtain that $\exists\ \xi = x_n$ between x_{n-1} and α such that $g^{(n)}(x_n) = 0$. Hence $f^{(n)}(\xi) = M$. □

Example 5.27. Show that for any $x \neq 0$ there is a real number y such that

$$e^x = 1 + x + \frac{e^y}{2}x^2.$$

In particular, we have $e^x > 1 + x$ for all $x \neq 0$.

Solution. Let $\alpha = 0$. By the Taylor's theorem, there is y between x and 0 such that

$$e^x = 1 + x + \frac{e^y}{2}x^2,$$

this implies $e^y = 2(e^x - 1 - x)/x^2$. Since $e^y > 0$ and $x^2 > 0$, we have $e^x > 1 + x$ for all $x \neq 0$. ∎

Example 5.28. Prove the inequality

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{for all } x \neq 0.$$

Solution. We have shown that for $f(x) = \ln(1+x)$, the Taylor's polynomial of degree n is

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n}.$$

By Taylor's theorem, $\exists\ \xi_1, \xi_2$ between x and 0 such that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{f'''(\xi_1)}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{f^{(4)}(\xi_2)}{4}x^4.$$

We have shown that

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

So $f'''(\xi_1) > 0$ and $f^{(4)}(\xi_2) < 0$. Hence the desired inequality follows. ∎

Example 5.29. Let $f : I \to \mathbb{R}$ be a differentiable function on an interval I. Suppose $f^{(n)}(t) = 0$ for all $t \in I$, where $n \in \mathbb{N}$, and there is $x_0 \in I$ such that

$$f(x_0) = f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0.$$

Prove that $f(x) = 0$ for all $x \in I$.

Solution. Let $x \in I$ and $x \neq x_0$.

Since $f^{(n)} = (f^{(n-1)})'$ exists on I, it follows that $f^{(n-1)}$ is continuous on the interval with endpoints x, x_0. By Taylor's theorem, there is ξ between x, x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n.$$

Since $f^{(n)}(\xi) = 0$ and $f^{(k)}(x_0) = 0$ for $k = 0, 1, \ldots, n-1$, we have $f(x) = 0$. ■

Example 5.30. Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function. Suppose f'' is bounded on $[0, \infty)$ and that

$$\lim_{x\to\infty} f(x) = A \in \mathbb{R}.$$

Prove that

$$\lim_{x\to\infty} f'(x) = 0.$$

Solution. Let $\varepsilon > 0$. We show that $\exists\, R > 0$ such that $|f'(x)| < \varepsilon$ for all $x > R$.

For each $x \in [0, \infty)$, by Taylor's theorem, there is ξ between x and $x+\varepsilon$ such that

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + \frac{f''(\xi)}{2!}\varepsilon^2.$$

Since $f(x) \to A$ as $x \to \infty$, $\exists\, R > 0$ such that

$$y > R \quad \Rightarrow \quad |f(y) - A| < \varepsilon^2.$$

By the triangle inequality, if $x > R$, we have

$$|f(x + \varepsilon) - f(x)| \leqslant |f(x + \varepsilon) - A| + |f(x) - A| < 2\varepsilon^2.$$

Since f'' is bounded, $\exists\, M > 0$ such that $|f''(x)| \leqslant M$ for all $x \in [0, \infty)$. In particular, $|f''(\xi)| \leqslant M$.

Now if $x > R$, we have

$$|f'(x)| = \frac{1}{\varepsilon}|f(x + \varepsilon) - f(x) - \frac{f''(\xi)}{2}\varepsilon^2| < (2 + M/2)\varepsilon.$$

This implies $\lim_{x\to\infty} f'(x) = 0$. ■

Exercise 5.2.

1. *State and prove the L'Hospital's rule 1 when* $a = -\infty$ *or* $L \in \{-\infty, \infty\}$.

2. *Prove the L'Hospital's rule 2 when* $a = -\infty$*:* $g'(x) \neq 0$, $\lim_{x\to-\infty} g(x) \in \{-\infty, \infty\}$, *and* $f'(x)/g'(x) \to \infty$ *as* $x \to a$.

3. *Prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is twice differentiable and* f'' *is continuous on* $\mathbb{R}$, *then*
$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

4. *Use the L'Hospital's rule to show that* $\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$. *You can use the properties of the exponential function from calculus.*

5. *Let* $f : (0,1) \to \mathbb{R}$ *and* $f^{(n)}$ *is defined on* $(0,1)$. *Assume* $\lim_{x\to 0} x^{2n} f^{(n)}(x) = L \in \mathbb{R}$. *Prove that* $\lim_{x\to 0} x^n f(x)$ *exists and calculate the limit.*

6. *Let* $f : [-1,1] \to \mathbb{R}$. *Assume* f''' *exists on* $[-1,1]$ *and*
$$f(-1) = 0, \quad f(0) = f'(0) = 0, \quad f(1) = 1.$$
Prove that there is $x_0 \in (-1,1)$ *such that* $f'''(x_0) \geqslant 3$.

■

Mathematical Analysis: Lecture 16

Sujin Khomrutai, Ph.D.

Summary. *In this lecture we study*

1. *The Riemann integral of a function defined on a closed interval.*
2. *Existence of Riemann integral for some bounded functions.*

6 Riemann Integration

In this chapter, we study the Riemann integral of a function. Throughout this chapter, we consider function defined on a closed interval $[a, b]$ and $a < b$.

6.1 The definition of Riemann Integral

A **partition** of an interval $[a, b]$ is a finite set of points

$$P = \{x_0, x_1, \ldots, x_N\} \quad \text{such that } a = x_0 < x_1 < \cdots < x_N = b.$$

If P is a partition, we denote

$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, N.$$

Definition 6.1 (Riemann integral). Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function and $P = \{x_1, \ldots, x_N\}$ a partition of $[a, b]$. We define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \quad (i = 1, 2, \ldots, N),$$

and consider the summation

$$U(f, P) = \sum_{i=1}^{N} M_i \Delta x_i, \quad L(f, P) = \sum_{i=1}^{N} m_i \Delta x_i.$$

They are called the **upper Riemann sum** and **lower Riemann sum**, respectively, with the partition P. Observe that $L(f, P) \leqslant U(f, P)$.

We define

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\},$$
$$\underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

They are called the **upper Riemann integral** and the **lower Riemann integral** of f over $[a,b]$, respectively.

f is said to be **Riemann integrable** provided the upper and lower Riemann integrals of f over $[a,b]$ are finite and they are equal. In that case, we denote the common value by

$$\int_a^b f(x)dx,$$

and it is called the **Riemann integral** of f over $[a,b]$.

Example 6.1. Show that $f(x) = k$, a constant, is Riemann integrable and

$$\int_a^b kdx = k(b-a).$$

Solution. For any partition $P = \{a = x_0 < x_1 < \cdots < x_N = b\}$ we have

$$U(f,P) = k(b-a) = L(f,P).$$

So $\overline{\int_a^b} kdx = k(b-a) = \underline{\int_a^b} kdx$, hence $f(x) = k$ is Riemann integrable and the integral is $\int_a^b kdx = k(b-a)$. ■

Example 6.2. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$. Is f is Riemann integrable over $[a,b]$?

Solution. Let $P = \{x_1, \ldots, x_N\}$ be a partition of $[a,b]$. We have $M_i = \sup_{[x_{i-1},x_i]} f(x) = 1$ and $m_i = \inf_{[x_{i-1},x_i]} f(x) = 0$. Then

$$U(f,P) = \sum_{i=1}^N M_i(x_i - x_{i-1}) = \sum_{i=1}^N 1 \cdot (x_i - x_{i-1}) = b-a,$$
$$L(f,P) = \sum_{i=1}^N m_i(x_i - x_{i-1}) = \sum_{i=1}^N 0 \cdot (x_i - x_{i-1}) = 0.$$

So it is clearly that the upper Riemann sum is $b-a > 0$ and the lower Riemann sum is 0. So f is not Riemann integrable. ■

Lemma 6.1. *The upper and lower Riemann integrals are finite.*

Proof. Since f is a bounded function, there are $m, M \in \mathbb{R}$ such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. In particular, $m \leqslant m_i \leqslant M_i \leqslant M$ for all $i = 1, \ldots, N$. Thus we get

$$m(b-a) \leqslant L(f, P) \leqslant U(f, P) \leqslant M(b-a).$$

This implies the upper and lower Riemann integrals are finite. □

Definition 6.2 (Refinements). A partion P' of $[a, b]$ is said to be a **refinement** of a partition P provided $P \subset P'$, that is every point in P is a point in P'.

Given two partitions P_1, P_2 of $[a, b]$, a **common refinement** is the partition $P' = P_1 \cup P_2$.

Lemma 6.2. *If P' is a refinement of P, then*

$$U(f, P') \leqslant U(f, P), \quad L(f, P') \geqslant L(f, P).$$

In particular, $U(f, P') - L(f, P') \leqslant U(f, P) - L(f, P)$.

Proof. It suffices to prove in the case that

$$U(f, P') \leqslant U(f, P), \quad L(f, P') \geqslant L(f, P),$$

when P' has one more point than P.

Let $P' = P \cup \{y\}$. Then $y \in (x_{i-1}, x_i)$ for some i, where $x_{i-1}, x_i \in P$. So

$$M_i' = \sup_{[x_{i-1}, y]} f(x) \leqslant M_i, \quad M_{i+1}' = \sup_{[y, x_i]} f(x) \leqslant M_i,$$

so

$$M_i' \Delta x_i' + M_{i+1}' \Delta x_{i+1}' \leqslant M_i \Delta x_i \quad \Rightarrow \quad U(f, P') \leqslant U(f, P).$$

Similarly, we have

$$m_i' \Delta x_i' + m_{i+1}' \Delta x_{i+1}' \geqslant m_i \Delta x_i \quad \Rightarrow \quad L(f, P') \geqslant L(f, P).$$

This proves the lemma. □

Lemma 6.3. $\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx.$

Proof. Let P_1, P_2 be arbitrary two partitions of $[a, b]$ and let P' be the common refinement of P_1, P_2. By the preceding lemma, we have

$$L(f, P_1) \leqslant L(f, P') \leqslant U(f, P') \leqslant U(f, P_2) \quad \Rightarrow \quad L(f, P_1) \leqslant U(f, P_2).$$

Keep P_2 fixed, and take the supremum over any P_1, we get

$$\underline{\int_a^b} f(x)dx \leqslant U(f, P_2).$$

This is true for any partition P_2, so by taking the infimum over all P_2, we get

$$\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx.$$

This proves the lemma. □

Remark. A useful fact is that

$$L(f, P_1) \leqslant U(f, P_2)$$

for any two partitions P_1, P_2 of $[a, b]$.

Theorem 6.1 (A necessary and sufficient condition for Riemann integrability). *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof. Homework. □

Example 6.3. If $f : [a, b] \to \mathbb{R}$ is a bounded Riemann integrable function over $[a, b]$ and $[c, d] \subset [a, b]$, then f is Riemann integrable over $[c, d]$.

Solution. Let $\varepsilon > 0$. Since f is integrable over $[a, b]$, $\exists$ a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. We can add c, d as points in P and the same inequality is still true. Consider the partition

$$P' = P \cap [c, d].$$

Then P' is a partition of $[c, d]$. Let $g = f|_{[c,d]}$. Then

$$U(g, P') - L(g, P') \leqslant U(f, P) - L(f, P) < \varepsilon.$$

So f is Riemann integrable over $[c, d]$. ■

Example 6.4. If $f : [a, b] \to \mathbb{R}$ is a bounded Riemann integrable function over $[a, b]$ and $[a, b] \subset [c, d]$, prove that the extension by zero $\tilde{f}$ of f to $[c, d]$ is Riemann integrable over $[c, d]$ and

$$\int_c^d \tilde{f}(x)dx = \int_a^b f(x)dx.$$

Solution. Let $\varepsilon > 0$. Since f is Riemann integrable over $[a, b]$, there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Consider the partition $P' = P \cup \{c, d\}$. It follows that

$$U(\tilde{f}, P') = U(f, P), \quad L(\tilde{f}, P') = L(f, P),$$

hence $U(\tilde{f}, P') - L(\tilde{f}, P') < \varepsilon$. So $\tilde{f}$ is Riemann integrable over $[c, d]$. Furthermore, by taking the infimum over all partition P, we conclude that

$$\int_a^b f(x)dx \leqslant \underline{\int_c^d} \tilde{f}(x)dx \leqslant \overline{\int_c^d} \tilde{f}(x)dx \leqslant \int_a^b f(x)dx.$$

Thus $\int_c^d \tilde{f}(x)dx = \int_a^b f(x)dx$. ■

Corollary 6.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose $a = c_1 < c_2 < \ldots, c_{k+1} = b$ and f is Riemann integrable over each $[c_j, c_{+1}]$. Then f is Riemann integrable over $[a, b]$ and*

$$\int_a^b f(x)dx = \sum_{j=1}^{k-1} \int_{c_j}^{c_{j+1}} f(x)dx.$$

Proof. Let $\varepsilon > 0$. For each j, let P_j be a partition for $[c_j, c_{j+1}]$ such that $U(f, P_j) - L(f, P_j) < \varepsilon/k$. Then the partition $P = P_1 \cup \cdots \cup P_k$ of $[a, b]$ satisfies

$$U(f, P) - L(f, P) = \sum_{j=1}^{k-1} (U(f, P_j) - L(f, P_j)) < \varepsilon.$$

This also implies $\sum_{j=1}^{k-1} U(f, P_j) = U(f, P) \to \int_a^b f(x)dx$ as $\varepsilon \to 0$. □

Theorem 6.2 (Every continuous function is Riemann integrable). *If f is a continuous function on $[a, b]$, then f is Riemann integrable over $[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f is continuous on a compact interval, f is uniformly continuous. Thus $\exists\, \delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon/(b - a).$$

Let N be a positive integer satisfying $(b - a)/N \leqslant \delta$. We choose a partition P to be $x_i = a + (i - 1)(b - a)/N$ for $i = 1, \ldots, N$. Then $\Delta x_i = (b - a)/N < \delta$. Since f is continuous, on each $[x_{i-1}, x_i]$, f attains the supremum and the infimum. Thus $\exists\, s, t \in [x_{i-1}, x_i]$ such that

$$M_i - m_i = f(s) - f(t) \quad \Rightarrow \quad M_i - m_i < \varepsilon/(b - a).$$

Consider

$$U(f, P) - L(f, P) = \sum_{i=1}^{N} (M_i - m_i)\Delta x_i < \varepsilon.$$

So we can conclude from the preceding theorem that f is Riemann integrable. □

Corollary 6.2. *Assume $f : [a, b] \to \mathbb{R}$ is a bounded piecewise continuous, i.e. f is continuous on every $x \in [a, b]$ except possibly at a finitely many points. Then f is Riemann integrable over $[a, b]$.*

Theorem 6.3. *If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is Riemann integrable over $[a, b]$.*

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that

$$(b - a)|f(b) - f(a)|/N \leqslant \varepsilon.$$

We define a partition P of $[a, b]$ by $x_i = a + (i - 1)(b - a)/N$ for $i = 1, 2, \ldots, N$. So $\Delta x_i = (b - a)/N$. Since f is monotonic, we have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{N} (M_i - m_i)\Delta x_i \\ &= \frac{b-a}{N} \sum_{i=1}^{N} (M_i - m_i) \\ &= \frac{b-a}{N} |f(b) - f(a)| \leqslant \varepsilon. \end{aligned}$$

So f is Riemann integrable. □

Example 6.5. Define

$$f(x) = 1 + \frac{1}{1 + 2^x} + \frac{1}{1 + 2^{2x}} + \cdots = \sum_{n=0}^{\infty} \frac{1}{1 + 2^{nx}} \quad (x > 0).$$

Prove that f is Riemann integrable on any $[1, 2]$.

Solution. Note that

$$|f(x)| \leqslant \sum_{n=0}^{\infty} \frac{1}{2^{nx}} = \frac{2^x}{2^x - 1} \leqslant 2 \quad (x \in [1, 2]).$$

Also, f is decreasing, so f is Riemann integrable over $[1, 2]$. ■

6.2 Properties of Riemann Integrals

Notation. For convenience, we introduce the notation

$$f \in \mathcal{R}[a, b] \quad \Leftrightarrow \quad f \text{ is bounded, Riemann integrable on } [a, b].$$

Theorem 6.4. *Let $f, g \in \mathcal{R}[a,b]$.*

1. $f+g, kf \in \mathcal{R}[a,b]$ *for any* $k \in \mathbb{R}$ *and*
$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \quad \int_a^b kf(x)dx = k\int_a^b f(x)dx.$$
2. *If* $f(x) \leqslant g(x)$ *for all* $x \in [a,b]$ *then*
$$\int_a^b f(x)dx \leqslant \int_a^b g(x)dx.$$
In particular, if $f(x) \geqslant 0$ *then* $\int_a^b f(x)dx \geqslant 0$.
3. *If* $c \in (a,b)$ *then* $f \in \mathcal{R}[a,c]$ *and* $f \in \mathcal{R}[c,b]$ *and*
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$
4. *If* $|f(x)| \leqslant M$ *for all* $x \in [a,b]$*, then*
$$\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx \leqslant M(b-a).$$

We shall apply the fact that if $f \in \mathcal{R}[a,b]$ then $\forall \varepsilon > 0$ $\exists$ a partition P of $[a,b]$ such that
$$U(f,P) \leqslant \int_a^b f(x)dx + \varepsilon, \quad L(f,P) \geqslant \int_a^b f(x)dx - \varepsilon.$$

Proof. 1. Since $\sup(f+g) \leqslant \sup f + \sup g$ and $\inf(f+g) \geqslant \inf f + \inf g$, it follows that for any partitions P_1, P_2 of $[a,b]$,
$$L(f,P_1) + L(g,P_1) \leqslant L(f+g,P_1) \leqslant U(f+g,P_2) \leqslant U(f,P_2) + U(g,P_2).$$
Since f, g are Riemann integrable on $[a,b]$, we can take P_2 so that $U(f,P_2) \leqslant \int_a^b f(x)dx + \varepsilon$ and $U(g,P_2) \leqslant \int_a^b g(x)dx + \varepsilon$, and, similarly, $\exists$ P_1 so that $L(f,P_1) \geqslant \int_a^b f(x)dx - \varepsilon, L(g,P_1) \geqslant \int_a^b g(x)dx - \varepsilon$. Thus
$$\begin{aligned}\int_a^b f(x)dx + \int_a^b g(x)dx - 2\varepsilon &\leqslant L(f+g,P_1) \\ &\leqslant U(f+g,P_2) \leqslant \int_a^b f(x)dx + \int_a^b g(x)dx + 2\varepsilon.\end{aligned}$$
Taking a common refinement, we conclude that $f+g \in \mathcal{R}[a,b]$ and the desired identity is true.

For kf, notice that

$$\begin{aligned} kL(f, P_1) &\leqslant L(kf, P_1) \leqslant U(kf, P_2) \leqslant kU(f, P_2) \quad \text{if } k \geqslant 0, \\ kU(f, P_1) &\leqslant L(kf, P_1) \leqslant U(kf, P_2) \leqslant kL(f, P_2) \quad \text{if } k < 0. \end{aligned}$$

If $k \geqslant 0$, we get $kf \in \mathcal{R}[a, b]$ and the desired identity is true, by the same technique as above.

For $k < 0$, we take P_2 so that $L(f, P_2) \geqslant \int_a^b f(x)dx - \varepsilon$ and P_1 so that $U(f, P_1) \leqslant \int_a^b f(x)dx + \varepsilon$. Then

$$k \int_a^b f(x)dx + k\varepsilon \leqslant L(kf, P_1) \leqslant U(kf, P_2) \leqslant k \int_a^b f(x)dx - k\varepsilon.$$

This also implies $kf \in \mathcal{R}[a, b]$ and the desired identity is true.

2. Since $f(x) \leqslant g(x)$, it follows that $U(f, P) \leqslant U(g, P)$ for any partition P. Taking the infimum over all P, we obtain $\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$.

3. That $f \in \mathcal{R}[a, c]$, $f \in \mathcal{R}[c, b]$ follow from Example 6.3. Define $f_1, f_2 : [a, b] \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} f(x) & x \in [a, c] \\ 0 & x \in (c, b] \end{cases}, \quad f_2(x) = \begin{cases} 0 & x \in [a, c) \\ f(x) & x \in [c, b]. \end{cases}$$

Observe that $f(x) = f_1(x) + f_2(x)$ for all $x \in [a, b]$. By Example 6.3 and 6.4, we get that $f_1, f_2 \in \mathcal{R}[a, b]$ and $\int_a^b f_1(x)dx = \int_a^c f(x)dx$, $\int_a^b f_2(x)dx = \int_c^b f(x)dx$, so the desired identity is true by part 1.

4. We have

$$f(x) \leqslant |f(x)| \leqslant M, \quad -M \leqslant -|f(x)| \leqslant f(x),$$

so by part 2, we get

$$\int_a^b f(x)dx \leqslant \int_a^b |f(x)|dx \leqslant M(b-a), \quad \int_a^b f(x)dx \geqslant -\int_a^b |f(x)dx \geqslant -M(b-a)$$

and the desired inequalities are true. □

Example 6.6. Prove that $f(x) = x \sin(1/x)$ is Riemann integrable on $[0, 1]$ and that

$$\int_0^1 x \sin \frac{1}{x} dx \leqslant \frac{1}{2}.$$

Solution. We note that $x \sin(1/x)$ is continuous at $x = 0$ by the squeeze theorem. So $f \in \mathcal{R}[0, 1]$. Now $|x \sin(1/x)| \leqslant x$ for all $x \in [0, 1]$, so

$$\int_0^1 x \sin \frac{1}{x} dx \leqslant \int_0^1 x dx = \frac{1}{2}.$$

■

We impose the following convention:

$$\int_a^a f(x)dx = 0, \qquad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

So we obtain

$$\int_a^b f(x)dx - \int_a^c f(x)dx = \int_c^b f(x)dx$$

for any arrangement of a, b, c provided the three terms are defined.

Example 6.7. Let $f\mathcal{R}[a,b]$. Define

$$F(x) = \int_a^x f(y)dy \quad (x \in [a,b]).$$

Prove that F is a continuous function.

Solution. We prove the continuity at $x_0 \in (a,b)$. The continuity at a, b will be left as exercise.

Let $\varepsilon > 0$. Since f is bounded, $\exists\ M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a,b]$. Consider

$$|F(x) - F(x_0)| = \left|\int_a^x f(x)dx - \int_a^{x_0} f(x)dx\right| = \left|\int_{x_0}^x f(x)dx\right|$$

So by the properties of integral, we get

$$|F(x) - F(x_0)| \leqslant M|x - x_0|.$$

Taking $\delta = \varepsilon/M$, it follows that if $|x - x_0| < \delta$ then $|F(x) - F(x_0)| < \varepsilon$, so F is continuous at x_0. ■

Exercise 6.1.

1. *Prove that if $f, g \in \mathcal{R}[a,b]$ then $fg, |f|, \max\{f,g\}, \min\{f,g\} \in \mathcal{R}[a,b]$.*
2. *Prove that if $f \in \mathcal{R}[a,b]$ and there is a constant $k > 0$ such that $f(x) \geqslant k$ for all $x \in [a,b]$ then $1/f \in \mathcal{R}[a,b]$.*
3. *Let f be a continuous function on $[a,b]$ and $f(x) < M$ for all $x \in [a,b]$. Prove that $\int_a^b f(x)dx < M(b-a)$.*
4. *If $f \geqslant 0, f \in \mathcal{R}[a,b]$ and $f(x_0) > 0$ for some $x_0 \in [a,b]$, prove that $\int_a^b f(x)dx > 0$.*

■

Mathematical Analysis: Lecture 17

Sujin Khomrutai, Ph.D.

Summary. *In this lecture, we study three important results for Riemann integral*

1. *The fundamental theorem of calculus.*
2. *The change of variables theorem.*
3. *The integration by parts formula.*

6.3 Three Important Results

Note that if $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) = 0$ for all $x \in (a, b)$, then g must be constant, by Corollary 5.1.

Definition 6.3 (Antiderivatives). Let $f : [a, b] \to \mathbb{R}$. A continuous function F on $[a, b]$ that satisfies $F'(x) = f(x)$ for all $x \in (a, b)$ is called an **antiderivative** of f.

If F_1 is also an antiderivative of f, then $(F_1 - F)'(x) = F_1'(x) - F'(x) = 0$ for all $x \in (a, b)$, hence $F_1 - F$ is constant. So the formula

$$\int f(x)dx = F(x) + C,$$

where C is an arbitrary constant, give all the antiderivatives of f. It is called the most general antiderivative of f or the **indefinite integral** of f.

Example 6.8. For $n \neq -1$, since $(x^{n+1}/(n+1))' = x^n$, $x^{n+1}/(n+1)$ is an antidervaitive of x^n. So

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$

Since $(-\cos x)' = \sin x$, $-\cos x$ is an antiderivative of $\sin x$. So

$$\int \sin x \, dx = -\cos x + C.$$

Since $(e^{kx}/k)' = e^{kx}$ $(k \neq 0)$, e^{kx}/k is an antiderivative of e^{kx}. So

$$\int e^{kx}\, dx = \frac{e^{kx}}{k} + C \quad (k \neq 0).$$

■

Lemma 6.4. *Let $f \in \mathcal{R}[a,b]$ and $x_0 \in (a,b)$. Define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(y)dy.$$

If f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

In particular, if f is a continuous function on $[a,b]$, then F is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$, i.e. F is an antiderivative of f.

Proof. Let $\varepsilon > 0$. Since f is continuous at x_0, $\exists\, \delta > 0$ such that if $x \in [a,b]$

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

If $0 < |x - x_0| < \delta$ then

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^x (f(y) - f(x_0))dy\right| < \varepsilon.$$

So F is differentiable and $F'(x_0) = f(x_0)$. □

Example 6.9. Consider $f(x) = 1/x$ for $x > 0$. Note that f is continuous on $(0,\infty)$, so for any $x > 0$, f is bounded and Riemann integrable from 1 to x. We define

$$\ln x = \int_1^x \frac{1}{s} ds.$$

Then $\ln : (0,\infty) \to \mathbb{R}$ is a differentiable function such that

$$(\ln x)' = \frac{1}{x} \quad \text{for all } x > 0.$$

It is called the **logarithmic function**.

The following arithmetic properties hold

$$\ln(xy) = \ln x + \ln y, \quad \ln\frac{x}{y} = \ln x - \ln y, \quad \ln(x^a) = a \ln x.$$

For instance, we have

$$\ln(xy) = \int_1^{xy} \frac{1}{s} ds = \int_1^x \frac{1}{s} ds + \int_x^{xy} \frac{1}{s} ds = \ln x + \int_1^y \frac{1}{t} dt = \ln x + \ln y,$$

where we have used the change of variables $t = xs$ in the second integral above. ■

Example 6.10. Since $(\ln x)' = 1/x > 0$, $\ln x$ is strictly increasing. In particular, the logarithmic function is one-to-one. The inverse of $\ln x$ is denoted by

$$e^x \quad \text{for } x \in \mathbb{R}.$$

So we have by definition

$$y = e^x \quad \Leftrightarrow \quad x = \ln y.$$

Thus $y > 0$ and $x \in \mathbb{R}$. It is called the **exponential function**.

By the chain rule, we have

$$x = \ln e^x \quad \Rightarrow \quad 1 = \frac{1}{e^x}(e^x)'$$

so

$$(e^x)' = e^x.$$

■

Example 6.11. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $f(x) \geqslant 0$ for all $x \in [a, b]$ and $\int_a^b f(x)dx = 0$. Prove that $f = 0$.

Solution. We define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(y)dy.$$

Let $x \in (a, b)$. Since $f \geqslant 0$, we have $\int_a^x f(y)dy \geqslant 0$ and $\int_x^b f(y)dy \geqslant 0$. So

$$0 \leqslant \int_a^x f(y)dy \leqslant \int_a^x f(y)dy + \int_x^b f(y)dy = \int_a^b f(y)dy = 0.$$

That is $F = 0$. This implies $F'(x) = 0 = f(x)$ for all $x \in (a, b)$. So $f = 0$ on $[a, b]$ by continuity. ■

Theorem 6.5 (**The fundamental theorem of calculus**). *Suppose a bounded Riemann integrable function f has an antiderivative F on $[a, b]$. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. We employ the mean value theorem. By definition, F is continuous on $[a, b]$ and differentiable on (a, b) with the derivative $F'(x) = f(x)$ for all $x \in (a, b)$.

Consider a partition $P = \{a = x_0 < x_1 < \cdots < x_N = b\}$ of $[a, b]$. By the mean value theorem, on each $[x_{i-1}, x_i]$ there is $\xi_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(\xi_i)\Delta x_i.$$

By telescoping, we get

$$F(b) - F(a) = \sum_{i=1}^{N} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{N} f(\xi_i)\Delta x_i.$$

Now $m_i \leqslant f(\xi_i) \leqslant M_i$, hence we get

$$L(f, P) \leqslant F(b) - F(a) \leqslant U(f, P).$$

For each $\varepsilon > 0$, it follows by Theorem 6.1 that

$$\int_a^b f(x)dx - \varepsilon \leqslant F(b) - F(a) \leqslant \int_a^b f(x)dx + \varepsilon.$$

Thus $F(b) - F(a) = \int_a^b f(x)dx$. □

The preceding theorem enables us in finding integrals by looking for antiderivative.

Example 6.12. Evaluate $\int_0^1 x^n dx$ for $n \in \mathbb{N}$.

Solution. Since

$$\left(\frac{x^{n+1}}{n+1}\right)' = x^n,$$

$x^{n+1}/(n+1)$ is an antiderivative of x^n on $[0, 1]$. By the fundamental theorem of calculus, we get

$$\int_0^1 x^n dx = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}.$$

■

A function g is called *continuously differentiable* on $[\alpha, \beta]$ if it is differentiable and g' is continuous on $[\alpha, \beta]$. We denote this by saying that $g \in C^1([\alpha, \beta])$.

Example 6.13 (A simple version of the change of variables theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $g \in C^1([\alpha, \beta])$ and $g(\alpha) = a, g(\beta) = b$. Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(\xi))g'(\xi)d\xi.$$

This means we can substitute $x = g(\xi)$ and change the variable of integration from x to ξ via the above formula.

Solution. Since f is continuous, $F(x) = \int_a^x f(y)dy$ is an antiderivative of f. Then $F'(x) = f(x)$ and $\int_a^b f(x)dx = F(b) - F(a)$. By the chain rule,

$$(F \circ g)'(\xi) = F'(g(\xi))g'(\xi) = f(g(\xi))g'(\xi)$$

So $F \circ g$ is an antiderivative of $(f \circ g)g'$. It follows by the fundamental theorem of calculus that

$$(F \circ g)(\beta) - (F \circ g)(\alpha) = \int_\alpha^\beta f(g(\xi))g'(\xi)d\xi.$$

Since $g(\beta) = b$ and $g(\alpha) = a$, it follows that the desired equality is true. ■

For most application, the result for the preceding example is enough. However, we can remove the continuity assumption of f in the preceding example.

Theorem 6.6 (**Change of variables**). *Let $f \in \mathcal{R}[a,b]$. Suppose $g \in C^1([\alpha,\beta])$, $g'(\xi) \neq 0$ for all $\xi \in [\alpha,\beta]$, and $g(\alpha) = a, g(\beta) = b$. Then $(f \circ g)g'$ is Riemann integrable on $[\alpha,\beta]$ and*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(\xi))g'(\xi)d\xi.$$

Proof. Since g' is continuous on $[\alpha,\beta]$, never zero, and $g(\alpha) < g(\beta)$, it follows that $g'(\xi) > 0$ for all $\xi \in [\alpha,\beta]$. By the chain rule, g is invertible with $g^{-1} \in C^1([a,b])$ and $g^{-1}(a) = \alpha, g^{-1}(b) = \beta$. So $\exists$ a constant $K > 0$ such that

$$|(g^{-1})'(x)| \leqslant K \quad \text{for all } x \in [a,b].$$

We prove that $f \circ g \in \mathcal{R}[\alpha,\beta]$. $f \circ g$ is clearly bounded. Let $\varepsilon > 0$. Since f is Riemann integrable, $\exists$ a partition $P = \{a = x_0 < x_1 < \cdots < x_N = b\}$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon/K.$$

Let $\tilde{P} = g^{-1}(P) = \{\alpha = \xi_0 < \xi_1 < \cdots < \xi_N = \beta\}$, a partition of $[\alpha,\beta]$ ($\xi_i = g^{-1}(x_i)$). By the mean value theorem, $\exists\ y_i \in (x_{i-1}, x_i)$ such that

$$\xi_i - \xi_{i-1} = (g^{-1})'(y_i)(x_i - x_{i-1}).$$

This implies $\Delta\xi_i \leqslant K\Delta x_i$. Then we have

$$U(f \circ g, \tilde{P}) - L(f \circ g, \tilde{P}) < \varepsilon.$$

Thus $f \circ g$ is Riemann integrable. Since g' is continuous on $[\alpha,\beta]$, it is bounded Riemann integrable on $[\alpha,\beta]$. Hence $(f \circ g)g' \in \mathcal{R}[\alpha,\beta]$.

Let $P, \tilde{P}$ be as above. We denote $h = (f \circ g)g'$. It was shown that h is Riemann integrable. By the mean value theorem, for each i, $\exists\ \eta_i \in (\xi_{i-1}, \xi_i)$ such that

$$x_i - x_{i-1} = g'(\eta_i)(\xi_i - \xi_{i-1}) \quad \Rightarrow \quad \Delta x_i = g'(\eta_i)\Delta\xi_i.$$

Let $M_i^* = \sup_{\xi \in [\xi_{i-1},\xi_i]} h(\xi)$ and $m_i^* = \inf_{\xi \in [\xi_{i-1},\xi_i]} h(\xi)$. Then we have

$$M_i^* := \sup_{\xi \in [\xi_{i-1},\xi_i]} h(\xi) \geqslant m_i g'(\eta_i) \quad (m_i = \inf_{\xi \in [\xi_{i-1},\xi_i]} (f \circ g)(\xi))$$

hence

$$U(h, \tilde{P}) = \sum_{i=1}^N M_i^* \Delta\xi_i \geqslant \sum_{i=1}^N m_i g'(\eta_i)\Delta\xi_i = L(f,P).$$

On the other hand,

$$m_i^* = \inf_{\xi \in [\xi_{i-1},\xi_i]} h(\xi) \leqslant M_i g'(\eta_i),$$

hence

$$L(h, \tilde{P}) = \sum_{i=1}^{N} m_i^* \Delta\xi_i \leqslant \sum_{i=1}^{N} M_i g'(\eta_i)\Delta\xi_i = U(f, P).$$

Now $U(h, \tilde{P}) \geqslant \int_a^b f(x)dx - \varepsilon$ and $L(h, \tilde{P}) \leqslant \int_a^b f(x)dx + \varepsilon$, hence

$$\int_a^b f(x)dx - \varepsilon \leqslant \int_\alpha^\beta h(\xi)d\xi \leqslant \int_a^b f(x)dx + \varepsilon$$

for all $\varepsilon > 0$. This implies the desired identity. □

Remark. In the preceding change of variable theorem, the assumption $g' \neq 0$ is required. This is in contrast to Example 6.13 where $g' \neq 0$ is not needed.

If $g(\alpha) = b$ and $g(\beta) = a$, then $g' < 0$ and in this case

$$\int_a^b f(x)dx = -\int_\alpha^\beta f(g(\xi))g'(\xi)d\xi = \int_\alpha^\beta f(g(\xi))|g'(\xi)|d\xi.$$

We can combine the two cases: Assuming $g([\alpha, \beta]) = [a, b]$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(\xi))|g'(\xi)|d\xi.$$

Example 6.14. Evaluate the integral

$$\int_0^1 \frac{e^x}{e^{2x} + 1} dx.$$

Solution. We put $f(x) = e^x/(e^{2x} + 1)$ and $x = g(\xi) = \ln \xi$, $1 < \xi < e$. Then $f \in \mathcal{R}[0, 1]$, $g \in C^1([1, e])$, and $g'(\xi) = 1/\xi > 0$ with $g(1) = 0, g(e) = 1$, so by the change of variable formula

$$\int_0^1 \frac{e^x}{e^{2x} + 1} dx = \int_1^e \frac{\xi}{\xi^2 + 1} \cdot \frac{1}{\xi} d\xi = \arctan e - \frac{\pi}{4}.$$

■

Example 6.15. Evaluate the integral

$$\int_a^b \frac{1}{x \ln x} dx \quad (e < a < b).$$

Solution. We put $f(x) = 1/(x \ln x)$ and $x = g(\xi) = e^\xi$, $\ln a < \xi < \ln b$. Then it is easy to check the conditions of the change of variable theorem, hence

$$\int_a^b \frac{1}{x \ln x} dx = \int_{\ln a}^{\ln b} \frac{1}{e^\xi \xi} \cdot e^\xi d\xi = \ln \frac{\ln b}{\ln a}.$$

■

Theorem 6.7. *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$ such that $f', g' \in \mathcal{R}[a, b]$. Then*

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)g'(x)dx.$$

Proof. We have $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$, that is $f(x)g(x)$ is an antiderivative of $f'(x)g(x) + f(x)g'(x)$. So by the fundamental theorem of calculus, we have

$$f(b)g(b) - f(a)g(a) = \int_a^b (f(x)g(x))'dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

This proves the theorem. □

Example 6.16. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^1 \int_0^x f(t)dtdx = \int_0^1 (1 - x)f(x)dx.$$

Solution. Let $F(x) = \int_0^x f(t)dt$. Then F is differentiable and $F'(x) = f(x)$. Integrating by parts yields

$$\int_0^1 \int_0^x f(t)dtdx = \int_0^1 F(x)dx = xF(x)\Big|_0^1 - \int_0^1 xF'(x)dx = F(1) - \int_0^1 xf(x)dx.$$

Since $F(1) = \int_0^1 f(x)dx$, we obtain the desired identity. ■

Example 6.17. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Assume $f(a) = f(b) = 0$ and

$$\int_a^b f(x)^2 dx = 1.$$

Prove that $\displaystyle\int_a^b xf(x)f'(x)dx = -1/2$.

Solution. We have

$$xf(x)f'(x) = \frac{1}{2}(xf(x)^2)' - \frac{1}{2}f(x)^2.$$

So integrating by parts yields

$$\begin{aligned}\int_a^b xf(x)f'(x)dx &= \frac{1}{2}\int_a^b (xf(x)^2)'dx - \frac{1}{2}\int_a^b f(x)^2 dx \\ &= \frac{1}{2}(bf(b)^2 - af(a)^2) - \frac{1}{2} = -\frac{1}{2}.\end{aligned}$$

■

Example 6.18. Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f'(x) > 0$ for all $x \in [a, b]$. Prove that

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a).$$

Solution. We write $f(x) = x' f(x)$, so by the integration by parts formula

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_a^b xf'(x)dx.$$

Since

$$xf'(x) = f^{-1}(f(x))f'(x) \quad \forall x \in [a, b],$$

it follows by the change of variables formula that

$$\int_a^b f^{-1}(f(x))f'(x)dx = \int_{f(a)}^{f(b)} f^{-1}(y)dy.$$

This together with the above identity conclude the desired result. ∎

Exercise 6.2.

1. *If* $f \in \mathcal{R}[a, b]$ *and* $x_0 \in [a, b]$, *evaluate the limit* $\displaystyle\lim_{x\to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} f(y)dy$.

2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous and periodic function with a period* $p > 0$. *Prove that* $\int_a^{a+p} f(x)dx = \int_0^p f(x)dx$.

3. *Let* $f : [0, \infty) \to \mathbb{R}$ *be continuous and* $\lim_{x\to\infty} f(x)$ *exist, and let* $a \in \mathbb{R}$. *Find* $\displaystyle\lim_{n\to\infty} \int_0^a f(nx)dx$. *(Hint. A change of variable and the L'Hospital's rule.)*

4. *Evaluate the integral* $\displaystyle\int_0^1 e^{\sqrt{x}}dx$.

5. *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable function such that* $f, f' \in \mathcal{R}[a, b]$. *Prove that*

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx dx = 0.$$

∎

Mathematical Analysis: Lecture 18

Sujin Khomrutai, Ph.D.

Summary. *In this chapter, we study sequences and series of functions.*

1. *Pointwise convergence*
2. *Uniform convergence*
3. *Some properties of uniform convergence*

7 Sequences and Series of Functions

In this chapter we study sequences and series of functions. We choose to consider mostly functions defined on an interval and having real (or complex) values to see the key results. There are various *modes* for a sequence (or series) of functions. For this chapter, we consider two most basic modes, the pointwise and the uniform convergences. Under the pointwise convergence, various properties will be lost when passing to the limit. The uniform convergence, however, preserves most of the properties.

7.1 Pointwise convergence

Definition 7.1. Let E be a non-empty set. A sequence of functions

$$\{f_n : E \to \mathbb{R}\}_{n=1}^{\infty} \quad \text{or simply } \{f_n\}$$

is said to **converge pointwise** provided for each $x_0 \in E$, the numerical sequence $\{f_n(x_0)\}_{n=1}^{\infty}$ is convergent.

If f_n converges pointwise, $\lim_{n\to\infty} f_n(x)$ at each x gives rise to a function:

$$f(x) := \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in E.$$

We denote this circumstance by saying that

$$f_n \to f \quad \text{pointwise on } E,$$

and f is called the **pointwise limit** of $\{f_n\}$.

Example 7.1 (Continuity). Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = x^n.$$

Find the pointwise limit function f. Determine whether f is a continuous function on $[0,1]$?

Solution. For $x_0 = 1$, we clearly have $f(1) = \lim_{n\to\infty} 1 = 1$. For $0 \leqslant x_0 < 1$, we have $f(x_0) = \lim_{n\to\infty} x_0^n = 0$. So

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \in [0,1). \end{cases}$$

Observe that all f_n are continuous function, but the pointwise limit f is discontinuous at $x = 1$. ■

Example 7.2 (Derivatives). Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}.$$

Find the pointwise limit function $f = \lim_{n\to\infty} f_n$. Compare the derivative $f'(0)$ and $\lim_{n\to\infty} f_n'(0)$.

Solution. By the squeeze theorem, we have

$$f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0.$$

So $f_n \to f = 0$ pointwise.

Next, the derivative $f'(0) = 0$ whereas

$$f_n'(0) = \sqrt{n} \to \infty.$$

So f_n' does not converge pointwise to f'. ■

Example 7.3 (Integrals). Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = nx(1-x^2)^n.$$

Find the pontwise limit function $f = \lim_{n\to\infty} f_n$. Compare the integrals $\int_0^1 f(x)dx$ and $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

Solution. If $x = 0$ or 1 we clearly have $f(x) = 0$. Next, for $0 < x < 1$ we have by the L'Hospital's rule that

$$f_n(x) = nx(1 - x^2)^n \to 0 \quad \text{as } n \to \infty.$$

So the limit function is $f(x) = 0$ for all $x \in [0, 1]$.

We explore the integrals. First, it is clearly that $\int_0^1 f(x)dx = 0$. For $n \in \mathbb{N}$ we have by a change of variable that

$$\int_0^1 f_n(x)dx = n \int_0^1 x(1 - x^2)^n dx = \frac{n}{2(n + 1)} \to \frac{1}{2}.$$

So $\int_0^1 f_n(x)dx$ does not converge to $\int_0^1 f(x)dx$ as $n \to \infty$. ■

Thus we have shown examples of $f_n \to f$ pointwise such that

1. f_n are all continuous, but f fails to be continuous.
2. $f'_n \not\to f'$ as $n \to \infty$.
3. $\int_a^b f_n(x)dx \not\to \int_a^b f(x)dx$ as $n \to \infty$.

Theorem 7.1. *Suppose $f_n \to f, g_n \to g$ pointwise on E. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function.*

1. *$f_n + g_n \to f + g$ pointwise on E.*
2. *$af_n \to af$ pontwise on E, for any constant α.*
3. *$f_n g_n \to fg$ pointwise on E.*
4. *If $g_n(x), g(x) \neq 0$ for all $x \in E$ then $f_n/g_n \to f/g$ pointwise on E.*
5. *$h \circ f_n \to h \circ f$ pointwise on E.*

Proof. By assumption, we have $f_n(x_0) \to f(x_0)$ and $g(x_0) \to g(x_0)$ for each $x_0 \in E$. Now by the corresponding properties for numerical sequences, we have

$$\begin{aligned}
&f_n(x_0) + g_n(x_0) \to f(x_0) + g(x_0), \\
&\alpha f_n(x_0) \to \alpha f(x_0), \\
&f_n(x_0)g_n(x_0) \to f(x_0)g(x_0), \\
&f_n(x_0)/g_n(x_0) \to f(x_0)/g(x_0), \\
&h(f_n(x_0)) \to h(f(x_0)).
\end{aligned}$$

So the theorem is proved. □

7.2 Uniform convergence

We have seen that the pointwise limit of a sequence of continuous functions may not be continuous. The following example gives the necessary and sufficient condition. It is the commutativity of taking limits in x and in n.

Example 7.4. Let f_n be functions on a metric space X which are continuous at x_0. Suppose $f_n \to f$ pointwise on X. Prove that f is continuous at $x_0 \in X$ if and only if

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Solution. It is immediate that

$$\lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right) = f(x_0).$$

On the other hand,

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{x\to x_0} f(x),$$

and hence f is continuous at $x_0 \in X$ if and only if

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = f(x_0) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

This proves the desired identity. ∎

By the triangle inequality, we have

$$|f(x) - f(x_0)| \leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|,$$

so if f_n can be arbitrarily close to f when n sufficiently large regardless of x, then the right hand side can be arbitrarily small provided x is close to x_0.

Definition 7.2 (Uniform convergence: Sequences). A sequence of functions $\{f_n\}$ is said to **converge uniformly** on E to a function f, written

$$f_n \to f \quad \text{uniformly on } E,$$

if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\text{if } n \geqslant N, \text{ then } |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E.$$

The function f, which is unique, is called the **uniform limit** of $\{f_n\}$.

Lemma 7.1. *If $f_n \to f$ uniformly on E, then $f_n \to f$ pointwise on E.*

Proof. Let $x_0 \in E$ and $\varepsilon > 0$. Since $f_n \to f$ uniformly on E, $\exists\, N \in \mathbb{N}$ such that

$$n \geqslant N \quad \Rightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in E.$$

This implies $|f_n(x_0) - f(x_0)| < \varepsilon$. So $f_n(x_0) \to f(x_0)$, hence $f_n \to f$ pointwise. □

We employ the following fact. Suppose there is a sequence of positive real numbers $a_n \to 0$ such that

$$|f_n(x) - f(x)| \leqslant a_n \quad \forall\, x \in E,$$

then $f_n \to f$ uniformly on E. On the other hand, if there is a constant $a > 0$ and a sequence $\{x_n\}$ in E such that

$$|f_n(x_n) - f(x_n)| \geqslant a,$$

then f_n does not converge uniformly to f.

Example 7.5. Determine whether $f_n = \dfrac{x}{nx+1}$ ($x \in [0,1]$) converges uniformly on $[0,1]$?

Solution. First we compute the pointwise limit. We have

$$f_n(x) = \frac{x}{nx+1} \to 0 \quad \text{pointwise on } [0,1].$$

So $f_n \to f = 0$ pointwise.

We have

$$|f_n(x) - f(x)| = \frac{x}{nx+1} \leqslant \frac{x}{nx} = \frac{1}{n} \quad \text{for all } x \in [0,1].$$

Thus $f_n \to 0$ uniformly on $[0,1]$. ∎

Example 7.6. Determine whether $f_n(x) = (\sin nx)/n$ ($x \in \mathbb{R}$) converges uniformly to any function?

Solution. Observe that $f_n(x) \to f(x) = 0$ for all $x \in \mathbb{R}$, by the squeeze theorem. So the pointwise limit function is $f = 0$.

Now, for any $x \in \mathbb{R}$, we have

$$|f_n(x) - f(x)| = \frac{|\sin nx|}{n} \leqslant \frac{1}{n}.$$

So $f_n \to 0$ uniformly on $\mathbb{R}$. ∎

Example 7.7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 1. Define $f_n(x) = f(1 + x/n)$. Prove that $\{f_n\}$ converges uniformly on $[0,1]$ and find the limit.

Solution. Since f is continuous, it follows that $f_n(x) = f(1+x/n) \to f(1)$ at each x. Since f is continuous at 1, we have $|f_n(x) - f(x)| = |f(1+x/n) - f(1)| < \varepsilon$ provided n is large enough. Thus $f_n \to f$ uniformly on $[0,1]$. ∎

Example 7.8. Determine whether $f_n = \dfrac{1}{nx+1}$ ($x \in [0,1]$) converges uniformly to any function?

Solution. Observe that

$$f_n = \frac{1}{nx+1} \to f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leqslant 1. \end{cases}$$

For each $n \in \mathbb{N}$, we have $f_n(1/n) = \frac{1}{2}$, so $|f_n(1/n) - f(1/n)| = 1/2$. This implies f_n does not converge uniformly to f. ∎

Theorem 7.2 (**Cauchy criterion**). *A sequence $\{f_n\}$ converges uniformly on E to a function f if and only if $\forall\,\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that*

$$m, n \geqslant N \quad \Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon \quad \textit{for all } x \in E.$$

Proof. ($\Rightarrow$) Suppose $f_n \to f$ uniformly on E. Let $\varepsilon > 0$. We can choose $N \in \mathbb{N}$ such that if $n \geqslant N$ then $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$, hence

$$|f_n(x) - f_m(x)| \leqslant |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon \quad \text{for all } x \in E.$$

($\Leftarrow$) The assumption implies $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$, so $f_n(x) \to f(x)$ for a function $f : E \to \mathbb{R}$. From $|f_n(x) - f_m(x)| < \varepsilon$, taking $m \to \infty$ (x fixed), we get

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in E.$$

Hence $f_n \to f$ uniformly on E. □

Definition 7.3. A sequence $\{f_n\}$ is said to be **uniformly bounded** (or **totally bounded**) if there is a constant $M > 0$ such that

$$|f_n(x)| \leqslant M \quad \text{for all } x \in E, n \in \mathbb{N}.$$

Theorem 7.3 (**Algebraic properties**). *Let $f_n \to f, g_n \to g$ uniformly.*

1. *$f_n + g_n \to f + g$ uniformly*
2. *$af_n \to af$ uniformly.*
3. *If $h : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $h \circ f_n \to h \circ f$ uniformly on E.*
4. *If $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}$ are uniformly bounded, then $f_n g_n \to fg$ uniformly.*
5. *If $\{1/f_n\}_{n=1}^{\infty}$ is uniformly bounded, then $1/f_n \to 1/f$ uniformly.*

Proof. First, $f_n \to f$ and $g_n \to g$ pointwise. So $f_n + g_n \to f + g$ and $\alpha f_n \to \alpha f$ pointwise.

1. Let $\varepsilon > 0$. Then $\exists\, N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/2$ and $|g_n(x) - g(x)| < \varepsilon/2$, hence

$$|(f_n + g_n)(x) - (f + g)(x)| \leqslant |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon.$$

So $f_n + g_n \to f + g$ uniformly.

2. Using the same n as above. Then

$$|(af_n)(x) - (af)(x)| = |a||f_n(x) - f(x)| < |a|\varepsilon \quad \text{for all } x \in E,$$

hence $\alpha f_n \to \alpha f$ uniformly.

For 3, 4, and 5 see Exercise 7.1. $\square$

Theorem 7.4 **(Dini's theorem).** *Let K be a compact subset of a metric space. Assume*

1. $f_n, f : K \to \mathbb{R}$ *are continuous for all* $n \in \mathbb{N}$,
2. $f_n \to f$ *pointwise on* K,
3. $f_n(x) \leqslant f_{n+1}(x)$ *(respectively,* $f_n(x) \geqslant f_{n+1}(x)$*) for all* $x \in K, n \in \mathbb{N}$.

Then $f_n \to f$ *uniformly on* K.

Proof. Let $\varepsilon > 0$. Define $g_n = f - f_n$ (resp., $g_n = f_n - f$, if f_n is decreasing), so $g_n(x) \geqslant 0$ for all $x \in K$ and $g_1 \geqslant g_2 \geqslant \cdots$. Let $A_n = \{x \in K : g_n(x) \geqslant \varepsilon\}$. Since g_n is continuous, A_n is compact. It is clearly that $A_{n+1} \subset A_n$ for all n.

Since $f_n(x_0) \to f(x_0)$ for all $x_0 \in K$, it follows that $g_n(x_0) \to 0$ as $n \to \infty$. Thus each $x_0 \notin A_n$ for all n sufficiently large. This implies

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

By the nested compact subsets theorem (3.10), then $A_N = \emptyset$ for some N. For such N, it follows that $g_N(x) < \varepsilon$ for all $x \in K$, hence $0 \leqslant f(x) - f_n(x) < \varepsilon$ for all $n \geqslant N$. Therefore $f_n \to f$ uniformly on K. $\square$

Exercise 7.1.

1. *Determine whether* $f_n(x) = \frac{1}{nx+1}$, $g_n(x) = \frac{x}{nx+1}$, $h_n(x) = \frac{nx}{n^2x^2+1}$ *converge uniformly on* $[0, 1]$*?*
2. *Let* $p, q > 0$. *Prove that the sequence of functions* $f_n(x) = \frac{x^p}{n+x^q}$ *converges uniformly on* $[0, \infty)$ *if and only if* $p < q$.
3. *Determine whether* $f_n(x) = \sin(x/n)$ *converges uniformly on* $\mathbb{R}$*?*
4. *Prove Theorem 7.3 3, 4, and 5.*

Mathematical Analysis: Lecture 19

Sujin Khomrutai, Ph.D.

Summary. *In this lecture we prove the following results:*

1. *the continuity,*
2. *the Riemann integrability, and*
3. *the differentiability*

of the uniform limit of a sequence of functions.

7.3 Properties of Uniform Convergence

Let E be an interval in $\mathbb{R}$.

Theorem 7.5 (Interchange of limits). *Let $f_n \to f$ uniformly on E and $x_0 \in E$. Suppose, for all n, $\lim_{x\to x_0} f_n(x)$ exist. Then $\lim_{x\to x_0} f(x)$ exists and*

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x).$$

In particular, if f_n is continuous at x_0, the so is f.

Proof. Suppose $a_n = \lim_{x\to x_0} f_n(x)$.

Since $f_n \to f$ uniformly, $\{f_n\}$ is uniformly Cauchy, i.e. for every ε, there is $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever $n, m \geqslant N$. Taking $x \to x_0$, we get $|a_n - a_m| < \varepsilon$, so $\{a_n\}$ is Cauchy in $\mathbb{R}$, hence $a_n \to a$ for some $a \in \mathbb{R}$.

Since $f_n \to f$ uniformly on E and $a_n \to a$, choose N such that $|f_N(x) - f(x)| < \varepsilon/3$ for all $x \in E$ and $|a_N - a| < \varepsilon/3$. Also, that $f_N(x) \to a_N$, we can choose $\delta > 0$ so that $|f_N(x) - a_N| < \varepsilon/3$ whenever $|x - x_0| < \delta$. If $|x - x_0| < \delta$ then

$$|f(x) - a| \leqslant |f(x) - f_N(x)| + |f_N(x) - a_N| + |a_N - a| < \varepsilon.$$

Thus $\lim_{x\to x_0} f(x) = a$. □

> **Corollary 7.1.** *The uniform limit of a sequence of continuous functions on E is continuous.*

Example 7.9. Prove that $f_n(x) = e^{-x/n}$ does not converge uniformly.

Solution. Note that $f_n(x) = f(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0, \end{cases}$ at each $x \in [0, 1]$. Since f is not continuous, so f_n does not converge uniformly (to f). ■

Example 7.10. Let $f_n : [a, b] \to \mathbb{R}$ be continuous functions and $f_n \to f$ uniformly on $[a, b]$. If $x_n \to x_0$ in $[a, b]$, prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$

Solution. We have f is continuous. Take $\delta > 0$ so that $|f(x)-f(x_0)| < \varepsilon/2$ whenever $|x - x_0| < \delta$. By the triangle inequality we have

$$|f_n(x_n) - f(x_0)| \leqslant |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \varepsilon,$$

by taking n large enough so that $|f_n(x) - f(x)| < \varepsilon/2$ for all x and $|x_n - x_0| < \delta$. ■

> **Theorem 7.6.** *Let $\{f_n\}$ be a sequence of bounded Riemann integrable functions on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$ and $\{f_n\}$ is a uniformly bounded, then f is a bounded Riemann integrable function on $[a, b]$, and, furthermore,*
>
> $$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Proof. Since $f_n \to f$ uniformly, choose n so that $|f_n(x) - f(x)| < \varepsilon/(b - a)$ for all $x \in [a, b]$. Thus we have

$$f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a} \quad \text{for all } x \in [a.b].$$

Since $f_n \in \mathcal{R}[a, b]$, we choose a partition P of $[a, b]$ so that

$$\int_a^b f_n(x)dx - \varepsilon \leqslant L(f_n, P) \leqslant U(f_n, P) \leqslant \int_a^b f_n(x)dx + \varepsilon$$

It follows that $L(f_n, P) - \varepsilon \leqslant L(f, P) \leqslant U(f, P) \leqslant U(f_n, P) + \varepsilon$, so

$$\int_a^b f_n(x)dx - 2\varepsilon \leqslant L(f, P) \leqslant U(f, P) \leqslant \int_a^b f_n(x)dx + 2\varepsilon.$$

So $U(f, P) - L(f, P) \leqslant 4\varepsilon$, therefore f is Riemann integrable.

Since $\{f_n\}$ is uniformly bounded, it follows that f is bounded. Thus $f \in \mathcal{R}[a,b]$. From that $|f_n(x) - f(x)| < \varepsilon/(b-a)$ we get

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leqslant \int_a^b |f_n(x) - f(x)|dx \leqslant \varepsilon,$$

therefore $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$. □

Example 7.11. Let $f_n(x) = (1 + x/n)^n$ for $x \in [0,1]$. Prove that $f_n \to e^x$ uniformly on $[0,1]$. Then calculate

$$\lim_{n\to\infty} \int_0^1 \left(1 + \frac{x}{n}\right)^n dx.$$

Solution. We use the L'Hospital's rule to get that

$$\left(1 + \frac{x}{n}\right)^n = \exp\{n \ln(1 + x/n)\} \to e^x$$

for each x. Thus $f_n \to f(x) = e^x$ pointwise.

By the mean value theorem, we have $e^x \geqslant 1 + x$ for all $x \geqslant 0$, so $1 + \frac{x}{n} \leqslant e^{x/n}$ or equivalently $(1 + \frac{x}{n})^n \leqslant e^x$. Let $g_n(x) = f(x) - f_n(x) = e^x - (1 + \frac{x}{n})^n$. For $x \geqslant 0$, we have

$$g_n'(x) = e^x - \left(1 + \frac{x}{n}\right)^{n-1} > e^x - \left(1 + \frac{x}{n}\right)^n \geqslant 0,$$

so g_n is strictly increasing. So $|f_n(x) - f(x)| \leqslant e - (1 + \frac{1}{n})^n \to 0$. We conclude that $f_n \to f$ uniformly on $[0,1]$. Also f_n is uniformly bounded by e.

By Theorem 7.6 we have

$$\lim_{n\to\infty} \int_0^1 \left(1 + \frac{x}{n}\right)^n dx = \int_0^1 e^x dx = e - 1.$$

■

Example 7.12. Let $f : [0,1/2] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^{1/2} f(x^n)dx = f(0)/2.$$

Solution. Let $f_n(x) = f(x^n)$ for $x \in [0,1/2]$. We have $x^n \to 0$ uniformly on $[0,1/2]$ (why?). So by the continuity of f, we get $f_n(x) = f(x^n) \to f(0)$ uniformly as well. Since f is continuous on $[0,1/2]$, it follows that f is bounded. So $\{f_n\}$ is uniformly bounded. Apply Theorem 7.6, we obtain

$$\int_0^{1/2} f_n(x)dx = \int_0^{1/2} f(x^n)dx \to \int_0^{1/2} f(0)dx = f(0)/2$$

as desired. ■

Theorem 7.7. *Let $\{f_n\}$ be a sequence of differentiable functions on (a,b). Suppose $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on (a,b) and there is $x_0 \in (a,b)$ such that $\{f_n(x_0)\}_{n=1}^{\infty}$ converges. Then there is a differentiable function $f:(a,b)\to\mathbb{R}$ such that*

$$f_n \to f \quad \text{uniformly on } (a,b) \quad \text{and} \quad f_n'(x)\to f'(x) \quad \text{for each } x\in(a,b).$$

Proof. First we show that $f_n \to f$ uniformly for a certain function $f:(a,b)\to\mathbb{R}$.

Let $\varepsilon > 0$. Since $f_n(x_0)$ is convergent and f_n' is uniformly convergent, $\exists\ N\in\mathbb{N}$ such that for $m,n \geqslant N$, we have

$$|f_n(x_0)-f_m(x_0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f_n'(x)-f_m'(x)| < \frac{\varepsilon}{2(b-a)} \quad \text{for all } x\in(a,b).$$

Fix such n,m. Applying the mean value theorem to the function $f_n - f_m$ we get that for any $x,y\in(a,b)$, $\exists\ \xi\in(a,b)$ such that

$$\begin{aligned} |(f_n(x)-f_m(x))-(f_n(y)-f_m(y))| &= |f_n'(\xi)-f_m'(\xi)||x-y| \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{$*$}$$

In particular, taking $y = x_0$, we get $|f_n(x)-f_m(x)| \leqslant |f_n(x_0)-f_m(x_0)| + \varepsilon/2 < \varepsilon$. This is true for all $x\in(a,b)$ (provided $n\geqslant N$), so $\exists$ a function $f:(a,b)\to\mathbb{R}$ such that $f_n\to f$ uniformly on (a,b).

Fix $\xi\in(a,b)$ and define the sequence

$$g_n(x) = \frac{f_n(x)-f_n(\xi)}{x-\xi} \quad (x\neq\xi).$$

By the inequality $(*)$ with $y=\xi$ we find that if $n\geqslant N$ then

$$|g_n(x)-g_m(x)| < \frac{\varepsilon}{2(b-a)} \quad \text{for all } x\neq\xi.$$

So $\{g_n\}$ converges uniformly on $(a,b)\setminus\{\xi\}$. Since $f_n\to f$ on (a,b), we get

$$g_n(x) \to \frac{f(x)-f(x_0)}{x-\xi} \quad \text{uniformly on } (a,b)\setminus\{\xi\}$$

Since f_n is differentiable at ξ, we have

$$\lim_{x\to\xi} g_n(x) = f_n'(\xi).$$

Applying Theorem 7.5 it follows that $\lim\limits_{x\to\xi}\dfrac{f(x)-f(\xi)}{x-\xi}$ exists, so f is differentiable at ξ, and

$$f'(\xi) = \lim_{n\to\infty} f_n'(\xi).$$

This is true for all $\xi\in(a,b)$. □

Example 7.13. Let $f_n : (a,b) \to \mathbb{R}$ be a sequence of functions. Suppose there is $c \in (a,b)$ such that $f_n(c) = 0$ for all n and f_n' converges uniformly on (a,b). Prove that f_n converges uniformly to a function f on (a,b) and that

$$\lim_{n\to\infty} f_n'(x) = f'(x) \quad \text{for all } x \in (a,b).$$

Solution. Since $f_n(c) = 0$ for all n, we have $f_n(c) \to 0$ as $n \to \infty$. Applying the preceding theorem with $x_0 = c$, we conclude that $\exists\ f : (a,b) \to \mathbb{R}$ such that $f_n \to f$ uniformly on (a,b) and $f_n'(x) \to f'(x)$ for all $x \in (a,b)$. ■

Example 7.14. Let f_n, f be bounded Riemann integrable functions on $[a,b]$. Suppose $f_n \to f$ uniformly on $[a,b]$. Prove that

$$\int_a^x f_n(y)dy \to \int_a^x f(y)dy \quad \text{uniformly on } [a,b].$$

Solution. Let $F_n(x) = \int_a^x f(y)dy$ and $F(x) = \int_a^x f(y)dy$. Then $F_n(a) \to F(a)$ and by the fundamental theorem of calculus

$$f_n(x) = F_n'(x), \quad f(x) = F'(x).$$

By assumption, $F_n' \to F'$ uniformly on (a,b), so we conclude by the above theorem that $F_n \to F$ uniformly on (a,b). Since F_n, F are continuous functions, it can be shown that $F_n \to F$ uniformly on $[a,b]$. ■

7.4 Series of Functions

By a series of functions on E, we mean

$$\sum_{n=1}^{\infty} f_n \quad \text{or simply } \sum f_n,$$

where $f_n : E \to \mathbb{R}$. It is understood as the sequence of functions $\{s_n\}$, where

$$s_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad (x \in E, n \in \mathbb{N}).$$

Of course, $\sum f_n \to f$ pointwise, means $s_n(x) \to f(x)$ for each $x \in E$, whereas, $\sum f_n \to f$ uniformly means, $s_n \to f$ uniformly on E.

Theorem 7.8 (Weierstrass M-test). *If $f_n : E \to \mathbb{R}$ satisfies $|f_n(x)| \leqslant M_n$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum f_n$ converges uniformly on E.*

Proof. For $n > m$, we have $|s_n(x) - s_m(x)| \leqslant \sum_{j=m+1}^{n} M_j \to 0$ as $n, m \to \infty$, so $\{s_n\}$ is uniformly Cauchy, hence $\{s_n\}$ converges uniformly. □

Example 7.15. Show that $\sum \dfrac{1}{x^2+n^2}$ converges uniformly on $\mathbb{R}$.

Solution. Let $f_n(x) = 1/(x^2+n^2)$. We have $|f_n(x)| \leqslant 1/n^2$ and $\sum 1/n^2$ conveges, so $\sum f_n$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-test. ■

Theorem 7.9. *Let $\{f_n\}$ be a sequence of functions on $[a, b]$.*

1. *If $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ and f_n are continuous functions, then f is a continuous function.*
2. *If $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ and f_n are uniformly bounded Riemann integrable functions over $[a, b]$, then f is bounded Riemann integrable over $[a, b]$ and*
$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$
3. *If f_n are differentiable on (a, b), $\sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in (a, b)$, and that the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly on (a, b), then $\sum_{n=1}^{\infty} f_n$ converges to a differentiable function $f : (a, b) \to \mathbb{R}$ uniformly on (a, b) and*
$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \text{for all } x \in (a, b).$$

Proof. Straightforward. □

Example 7.16. Let $\{c_n\}$ be a sequence of real numbers such that $\sum |c_n|$ converges. Define $f(x) = \sum_{n=1}^{\infty} c_n \sin nx$ $(x \in \mathbb{R})$. Find $\int_0^{\pi} f(x)dx$.

Solution. By the Weierstrass M-test, $f(x)$ is well-defined because the series $\sum c_n \sin nx$ converges uniformly on any interval $[a, b]$. So
$$\int_0^{\pi} f(x)dx = \sum_{n=1}^{\infty} \int_0^{\pi} c_n \sin nx dx = \sum_{n=1}^{\infty} c_n \frac{1 + (-1)^{n+1}}{n}.$$
■

Example 7.17. Show that $f(x) = \sum \frac{x^n}{n!}$ $(x \in \mathbb{R})$ is a differentiable function and $f'(x) = f(x)$. (This function is the exponential function e^x.)

Solution. Fix $R > 0$. Let $f_n(x) = x^n/n!$. Consider on $(-R, R)$. Clearly, $\sum f_n(0)$ converges and $\sum f_n'(x) = \sum f_n(x)$ converges uniformly on $(-R, R)$ by the Weiertrass M-test. Therefore $f(x) = \sum x^n/n!$ converges uniformly on $(-R, R)$ and $f'(x) = \sum f_n'(x) = \sum f_n(x) = f(x)$. ■

Exercise 7.2.

1. *Find* $\lim_{n\to\infty} \int_0^1 e^{-x^2/n} dx$.

2. *If* $f : [0,1] \to \mathbb{R}$ *is a continuous function, evaluate* $\lim_{n\to\infty} \int_0^1 f(x/n)dx$.

3. *If* f_n *are twice differentiable functions on* (a,b), $\exists\ x_0, x_1 \in (a,b)$ *such that* $\{f_n(x_0)\}_{n=1}^{\infty}$, $\{f_n'(x_1)\}_{n=1}^{\infty}$ *converge, and* f_n'' *converge uniformly on* (a,b)*, then* $\exists$ *a twice differentiable function* $f : (a,b) \to \mathbb{R}$ *such that*

$$f_n \to f, \quad f_n' \to f', \quad f_n'' \to f' \quad \textit{uniformly on } (a,b).$$

4. *Prove that* $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ *is a differentiable function on* $\mathbb{R}$.

Mathematical Analysis: Lecture 20

Sujin Khomrutai, Ph.D.

7.5 Arzelà-Ascoli and Stone-Weierstrass Theorems

Summary. *In this lecture we study*

1. *the Arzelà-Ascoli theorem.*
2. *the Weierstrass approximation theorem, and*
3. *the Stone-Weierstrass theorem.*

7.5.1 Arzelà-Ascoli Theorem

We begin with the following definitions.

Definition 7.4. A sequence $\{f_n : E \to \mathbb{R}\}$ is said to be **pointwise bounded** if for each $x_0 \in E$ the sequence of numbers $\{f_n(x_0)\}$ is bounded. Equivalently, $\{f_n\}$ is pointwise bounded if and only if there is a function $\varphi : E \to \mathbb{R}$ such that $|f_n(x)| \leqslant \varphi(x)$ for all $n \in \mathbb{N}$.

Definition 7.5. A sequence $f_n : [a, b] \to \mathbb{R}$ is said to be **equicontinuous** provided for each $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in E$ and $|x-y| < \delta$ then $|f_n(x)-f_n(y)|$ for all $n \in \mathbb{N}$.

Let us prove the following Cauchy diagonal argument.

Lemma 7.2. *Suppose $\{a_{n,m}\}_{n,m=1}^{\infty}$ is a double sequence of real numbers such that for each m fixed, the sequence $\{a_{n,m}\}_{n=1}^{\infty}$ is bounded. Then there is a sequence of positive integers*

$$n_1 < n_2 < \cdots < n_k < \cdots$$

such that for each $m \in \mathbb{N}$, the sequence $\{a_{n_k,m}\}_{k=1}^{\infty}$ is convergence.

Proof. We shall repeatedly employed the Heine-Borel theorem.

For $m = 1$, since $\{a_{n,1}\}_{n=1}^{\infty}$ is bounded, we can choose $n_1^1 < n_2^1 < \cdots < n_k^1 < \cdots$ such that $\{a_{n_k^1,1}\}_{k=1}^{\infty}$ is convergence.

For $m = 2$, the sequence $\{a_{n_k^1,2}\}_{k=1}^{\infty}$ is bounded, so there is a subsequence of $\{n_k^1\}_{k=1}^{\infty}$ denoted by $\{n_k^2\}_{k=1}^{\infty}$ such that $\{a_{n_k^2,2}\}_{k=1}^{\infty}$ is convergence.

We continue inductively to get for each $m > 2$, a subsequence $\{n_k^m\}_{k=1}^{\infty}$ of $\{n_k^{m-1}\}_{k=1}^{\infty}$ such that $\{a_{n_k^m,m}\}_{k=1}^{\infty}$ is convergence.

Observe that if $m' \geqslant m$ then every $n_\ell^{m'}$ $(\ell \in \mathbb{N})$ lies in the sequence $\{n_k^m\}_{k=1}^{\infty}$.

Let $n_k = n_k^k$ for all $k \in \mathbb{N}$. For each $m \in \mathbb{N}$, $\{a_{n_k,m}\}_{k=n}^{\infty}$ as a subsequence of the convergence sequence $\{a_{n_k^m,m}\}_{k=1}^{\infty}$ is convergence. Therefore $\{a_{n_k,m}\}_{k=1}^{\infty}$ is convergence for all m. □

Corollary 7.2. *Let C be an infinite countable set. Suppose $f_n : C \to \mathbb{R}$ is a sequence of functions such that $\{f_n(x)\}_{n=1}^{\infty}$ is bounded for each $x \in C$. Then there is a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ is convergence for all $x \in C$.*

Proof. Denote $C = \{x_m : m \in \mathbb{N}\}$ and $a_{n,m} = f_n(x_m)$. The result follows by the preceding lemma. □

Theorem 7.10 (**Arzelà-Ascoli theorem**)**.** *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions. Suppose $\{f_n\}$ satisfies the following two conditions:*

1. *$\{f_n\}$ is pointwise bounded, and*
2. *$\{f_n\}$ is equicontinuous.*

Then $\{f_n\}$ is uniformly bounded and there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that converges uniformly to a function f on $[a,b]$.

Proof. In the following proof, x, y are assumed to be in $[a,b]$.

Take a countable dense subset C in $[a,b]$. Let $\varepsilon > 0$. Since $\{f_n\}$ is equicontinuous, there is $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon \quad \text{for all } n.$$

Since $[a,b]$ is compact and C is dense in $[a,b]$, there are $x_1, \ldots, x_r \in C$ such that $\{B_\delta(x_i)\}_{i=1}^{r}$ covers $[a,b]$. Let $M_i > 0$ be such that $|f_n(x_i)| \leqslant M_i$ for all n and $M = \max\{M_1, \ldots, M_r\}$. Now for any x, choose x_i so that $|x - x_i| < \delta$, hence

$$|f_n(x)| \leqslant |f_n(x) - f_n(x_i)| + |f_n(x_i)| \leqslant M + \varepsilon.$$

It follows that $\{f_n\}$ is uniformly bounded.

For each $x \in C$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded by the assumption, so we get by the preceding corollary that there is a subsequence $\{f_{n_k}\}$ that converges at each point in C.

Now it suffices to show that $\{f_{n_k}\}$ is uniformly Cauchy on $[a, b]$.

Let ε, δ be as above. For $x \in [a, b]$, choose $y \in C$ such that $|x - y| < \delta$. Then

$$|f_{n_k}(x) - f_{n_m}(x)| \leqslant |f_{n_k}(x) - f_{n_k}(y)| + |f_{n_k}(y) - f_{n_m}(y)| + |f_{n_m}(y) - f_{n_m}(x)| < 3\varepsilon,$$

by taking n, m sufficiently large ($\{f_{n_k}(y)\}_{k=1}^{\infty}$ is convergent). This is true for any $x \in [a, b]$, so $\{f_{n_k}\}$ is uniformly Cauchy, hence it converges uniformly on $[a, b]$. $\square$

Remark. *The Arzelà-Ascoli theorem can be generalized from $[a, b]$ to any compact metric space K. In fact, K has a countable dense subset by considering open covering $\{B_{1/i}(x) : x \in K\}$ for each $i \in \mathbb{N}$.*

Example 7.18. Let $u_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions satisfying $u_n(x) \geqslant 0$ for all n and

$$u_n(x) = \alpha + \int_0^x e^{-u_n(y))} dy \quad (x \in [a, b], n \in \mathbb{N}),$$

where $\alpha \geqslant 0$ is a constant. Show that there is a subsequence $\{n_k\}$ such that u_{n_k} converges uniformly to a function.

Solution. By the triangle inequality, we have

$$|u_n(x)| \leqslant \alpha + \int_a^x e^{-u_n(y)} dy \leqslant \alpha + b \quad \forall\, x \in [a, b], n \in \mathbb{N}.$$

So $\{u_n\}$ is uniformly bounded. For $x_1, x_2 \in [a, b]$, we get by subtraction

$$\begin{aligned} |u_n(x_1) - u_n(x_2)| &= \left| \int_a^{x_1} e^{-u_n(y)} dy - \int_a^{x_2} e^{-u_n(y)} dy \right| \\ &\leqslant \left| \int_{x_2}^{x_1} e^{-u_n(y)} dy \right| \leqslant |x_2 - x_1|, \end{aligned}$$

so $\{u_n\}$ is equicontinuous. The conclusion follows from the Arzelà-Ascoli theorem. ■

7.5.2 Weierstrass Theorem

We begin with the following basic version of approximation theorem.

Theorem 7.11 (Weierstrass theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for each $\varepsilon > 0$ there is a polynomial p (with coefficients in $\mathbb{R}$) such that*

$$|f(x) - p(x)| < \varepsilon \quad \text{for all } x \in [a, b].$$

In other words, there is a sequence $\{p_n\}$ *of polynomials such that*

$$p_n \to f \quad \text{uniformly on } [a, b].$$

Proof. Without loss of generality, assume $a = 0, b = 1$. We use Bernstein polynomials $\binom{N}{k} x^k (1-x)^{N-k}$. Consider

$$p_N(x) = \sum_{k=0}^{N} f(k/N) \binom{N}{k} x^k (1-x)^{N-k}.$$

Observe that $f(x) = \sum_{k=0}^{N} f(x) \binom{N}{k} x^k (1-x)^{N-k}$ by the binomial theorem. So

$$|f(x) - p_N(x)| \leqslant \sum_{k=0}^{N} |f(x) - f(k/N)| \binom{N}{k} x^k (1-x)^{N-k}.$$

Since f is uniformly continuous on $[a, b]$, $\exists\ \delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon/2$. We split the summation over k above into $\{k : |x - k/N| < \delta\}$ and $\{k : |x - k/N| \geqslant \delta\}$. For the first case, we get

$$\sum_{\{k:|x-k/N|<\delta\}} |f(x) - f(k/N)| \binom{N}{k} x^k (1-x)^{N-k} \leqslant \varepsilon/2.$$

Since f is continuous on $[a, b]$, $\exists\ M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a, b]$. Also,

$$\begin{aligned} &\sum_{\{k:|x-k/N|\geqslant\delta\}} |f(x) - f(k/N)| \binom{N}{k} x^k (1-x)^{N-k} \\ &\leqslant \frac{2M}{\delta^2} \sum_{k=0}^{N} (x - \frac{k}{N})^2 \binom{N}{k} x^k (1-x)^{N-k} \\ &= \frac{2M}{\delta^2} \frac{1}{N} x(1-x) \leqslant \frac{M}{2N\delta^2}. \end{aligned}$$

Taking N large enough so that $M/(2N\delta^2) < \varepsilon/2$ and we are done. □

Example 7.19. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that for every $\varepsilon > 0$, there is a polynomial $p(x)$ such that $\left| \int_a^b (f(x) - p(x))dx \right| < \varepsilon$ ■

Example 7.20. Prove that if $\varepsilon > 0$, there is a polynomial P such that $P(0) = P(\pi) = 0$ and $|P(x) - \sin x| \leqslant \varepsilon$ for all $x \in [0, \pi]$.

Solution. Firstly, we apply the Weierstrass theorem to get a polynomial Q such that $|Q(x) - \sin x| \leqslant \varepsilon/3$. Let

$$P(x) = Q(x) - Q(0)\frac{1}{\pi}(\pi - x) - Q(\pi)\frac{1}{\pi}x.$$

Then P is a polynomial and $P(0) = P(\pi) = 0$.

Since $\sin 0 = \sin \pi = 0$, we have by the inequality $|Q(x) - \sin x| \leqslant \varepsilon/3$ that

$$|Q(0)| \leqslant \varepsilon/3, \quad |Q(\pi)| \leqslant \varepsilon/3.$$

By the triangle inequality, it follows that

$$|P(x) - \sin x| \leqslant |Q(x) - \sin x| + |Q(0)| + |Q(\pi)| \leqslant \varepsilon.$$

Thus P has the desired properties. ■

Example 7.21. Let $f : [-a, a] \to \mathbb{R}$ be an odd continuous function and $\varepsilon > 0$. Prove that there is an odd polynomial P such that $\|f - P\|_\infty \leqslant \varepsilon$.

Solution. Note that $f(-x) = -f(x)$ for all $x \in [-a, a]$. Apply the Weierstrass theorem, we get a polynomial Q such that $\|f - Q\|_\infty \leqslant \varepsilon/2$. Let

$$P(x) = Q(x) - Q(-x).$$

Then $P(-x) = -P(x)$ for all $x \in \mathbb{R}$, hence P is an odd polynomial. Next, by the triangle inequality,

$$|f(x) - P(x)| \leqslant |f(x) - Q(x)| + |f(x) + Q(-x)|.$$

Since $|f(x) - Q(x)| \leqslant \|f - Q\|_\infty \leqslant \varepsilon/2$ and $|f(x) + Q(-x)| = |-f(-x) + Q(-x)| \leqslant \|f - Q\|_\infty \leqslant \varepsilon/2$, it follows that $\|f - P\|_\infty \leqslant \varepsilon$. So P has the desired property. ■

Example 7.22. Let $f \in C([a, b])$. Assume that

$$\int_a^b x^n f(x)dx = 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Prove that $f = 0$.

Solution. If $P(x) = a_n x^n + \cdots + a_1 x + a_0$ then by the assumption, we have

$$\int_a^b P(x)f(x)dx = 0.$$

Thus, the above identity holds for any polynomial P. Let $M = \int_a^b f(x)^2 dx$.

Suppose to get a contradiction that $M > 0$. By the Weierstrass theorem, $\exists$ a polynomial P such that $\|P - f\|_\infty \leqslant M/2(b - a)$. Thus

$$\int_a^b (P(x) - f(x))^2 dx \leqslant \|P - f\|_\infty (b - a) \leqslant \frac{M}{2}.$$

On the other hand, since $\int_a^b P(x)f(x)dx = 0$, we have

$$\int_a^b (P(x) - f(x))^2 dx = \int_a^b (P(x)^2 + f(x)^2)dx \geqslant M.$$

This contradiction implies $M = 0$, hence $f = 0$. ■

7.5.3 The Stone-Weierstrass theorem

Let

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}.$$

It is a metric space with the metric

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

We also put

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|, \quad \text{so} \quad d(f,g) = \|f-g\|_\infty.$$

It is easy to see that $C([a,b])$ is a vector space. So we can compute $f+g \in C([a,b])$ and $\alpha f \in C([a,b])$ for any $f,g \in C([a,b])$, $\alpha \in \mathbb{R}$. Furthermore, we can multiply fg, so, in fact, $C([a,b])$ is an **algebra** over $\mathbb{R}$ in the following sense.

Definition 7.6. A collection $\mathcal{A}$ of real valued functions defined on a set $S \neq \emptyset$ is called an **algebra** over $\mathbb{R}$ if $\mathcal{A}$ is closed under addition, scalar multiplication, and multiplication in $\mathcal{A}$, i.e. if $f,g \in \mathcal{A}$ and $\alpha \in \mathbb{R}$ then

$$f+g,\ \alpha f,\ fg \in \mathcal{A}.$$

If $\mathcal{A}$ is an algebra of functions on S, a subset $\mathcal{B} \subset \mathcal{A}$ is called a **subalgebra** provided it is closed the addition, multiplication, and scalar multiplication.

Example 7.23. Prove that $C([a,b])$ is an algebra over $\mathbb{R}$ and that $\mathcal{P}$, the set of all polynomials (with real coefficients), is a subalgebra of $C([a,b])$.

Solution. If f,g are continuous functions on $[a,b]$ and $\alpha \in \mathbb{R}$ then $f+g, \alpha f, f\cdot g$ are continuous on $[a,b]$ as well. So $C([a,b])$ is an algebra over $\mathbb{R}$.

If $P,Q \in \mathcal{P}$ and $\alpha \in \mathbb{R}$ then clearly

$$P+Q, \quad \alpha P, \quad P\cdot Q \quad \text{are polynomials.}$$

So $\mathcal{P}([a,b])$ is a subalgebra of $C([a,b])$. ∎

Example 7.24. Let $x_0 \in [a,b]$ and define

$$\mathcal{F}(x_0) = \{f \in C([a,b]) : f(x_0) = 0\}.$$

Show that $\mathcal{F}(x_0)$ is a subalgebra of $C([a,b])$.

Solution. If $f,g \in \mathcal{F}(x_0)$ and $\alpha \in \mathbb{R}$ then clearly

$$(f+g)(x_0) = f(x_0)+g(x_0) = 0,\ (\alpha f)(x_0) = \alpha f(x_0) = 0,\ (f\cdot g)(x_0) = f(x_0)g(x_0) = 0,$$

so

$$f+g,\ \alpha f,\ f\cdot g\ \in \mathcal{F}(x_0).$$

Thus $\mathcal{F}(x_0)$ is a subalgebra of $C([a,b])$. ∎

Definition 7.7. A set of functions S is said to **separate points** in $[a,b]$ if for any $x \neq y$, there is $f \in S$ such that

$$f(x) \neq f(y).$$

For $A, B \subset C([a,b])$, A is said to be **dense** in B provided for each $g \in B$, if $\varepsilon > 0$ then $\exists\, f \in A$ such that

$$\|f - g\|_\infty < \varepsilon.$$

Theorem 7.12 (Stone-Weierstrass theorem). *Let $\mathcal{A}$ be a subalgebra of $C([a,b])$. Suppose $\mathcal{A}$ separates points in $[a,b]$ and it contains the constant functions. Then $\mathcal{A}$ is dense in $C([a,b])$.*

Proof. Let

$$\mathcal{B} = \{g \in C([a,b]) : \forall\, \varepsilon > 0\, \exists\, f \in \mathcal{A} \text{ such that } \|f - g\|_\infty \leqslant \varepsilon\}.$$

We have to show that $\mathcal{B} = C([a,b])$. Clearly $\mathcal{A} \subset \mathcal{B}$.

Step 1. $\mathcal{B}$ is a subalgebra of $C([a,b])$.

Let $\varepsilon > 0$. If $g, \tilde{g} \in \mathcal{B}$ then $\exists\, f, \tilde{f} \in \mathcal{A}$ such that $\|f - g\|_\infty \leqslant \varepsilon/2$ and $\|\tilde{f} - \tilde{g}\|_\infty \leqslant \varepsilon/2$. By the triangle inequality we get

$$\|(f + \tilde{f}) - (g + \tilde{g})\|_\infty \leqslant \|f - g\|_\infty + \|\tilde{f} - \tilde{g}\|_\infty \leqslant \varepsilon.$$

So $g + \tilde{g} \in \mathcal{B}$. Since $0 \in \mathcal{A} \subset \mathcal{B}$, we have $0 \cdot g \in \mathcal{B}$. Now if $\alpha \neq 0$, $\exists\, f \in \mathcal{B}$ such that $\|f - g\|_\infty \leqslant \varepsilon/|\alpha|$. Thus

$$\|(\alpha f) - (\alpha g)\|_\infty \leqslant \varepsilon.$$

This means $\alpha g \in \mathcal{B}$ for all $\alpha \in \mathbb{R}$. It is left as an exercise to show that $g \cdot \tilde{g} \in \mathcal{B}$.

Step 2. If $g \in \mathcal{B}$ then $|g| \in \mathcal{B}$.

Let $\|g\|_\infty = M$ and $\varepsilon > 0$. Apply the Weierstrass' theorem to the continuous function $h(x) = |x|$ $(x \in [-M, M])$, there is a polynomial P such that $\|P - h\|_\infty \leqslant \varepsilon$. Since $\mathcal{A}$ is an algebra, we have $P \circ g \in \mathcal{A}$. Now

$$\|P \circ g - |g|\|_\infty = \sup_{x \in [a,b]} |P(g(x)) - |g(x)|| \leqslant \|P - h\|_\infty \leqslant \varepsilon.$$

Thus $|g| \in \mathcal{B}$ as needed.

Step 3. If $g_1, \ldots, g_n \in \mathcal{B}$ then $\max\{g_1, \ldots, g_n\}, \min\{g_1, \ldots, g_n\} \in \mathcal{B}$.

Note that

$$\max\{g_1, \ldots, g_n\}(x) = \max\{g_1(x), \ldots, g_n(x)\} \quad \text{for all } x \in [a,b].$$

Observe that $\max\{g_1, g_2\} = \frac{1}{2}(g_1 + g_2 + |g_1 - g_2|)$. Now if $g_1, g_2 \in \mathcal{B}$, it follows that $\frac{1}{2}(g_1 + g_2) \in \mathcal{B}$ since $\mathcal{B}$ is a subalgebra, and $|g_1 - g_2| \in \mathcal{B}$ by **Step 2**, hence

$\max\{g_1, g_2\} \in \mathcal{B}$. By induction, we have $\max\{g_1, \ldots, g_n\} \in \mathcal{B}$. The minimum can be proved by a similar argument and that $\min\{g_1, g_2\} = \frac{1}{2}(g_1 + g_2 - |g_1 - g_2|)$.

Step 4. If $f \in C([a, b])$ and $\xi \neq \eta$ are distinct points in $[a, b]$, $\exists\ g_{\xi\eta} \in \mathcal{A}$ such that $g_{\xi\eta}(\xi) = f(\xi)$ and $g_{\xi\eta}(\eta) = f(\eta)$.

Since $\mathcal{A}$ separates points in $[a, b]$, $\exists\ h \in \mathcal{A}$ such that $h(\xi) \neq h(\eta)$. Then define

$$g_{\xi\eta}(x) = f(\xi) + \frac{f(\xi) - f(\eta)}{h(\xi) - h(\eta)}(h(x) - h(\xi)) \quad \text{for } x \in [a, b].$$

Since $\mathcal{A}$ is a subalgebra of $C([a, b])$ and constant functions are elements of $\mathcal{A}$, it follows that $g_{\xi\eta} \in \mathcal{A}$. Clearly, $g_{\xi\eta}$ has the desired properties.

Step 5. If $f \in C([a, b])$, $\xi \in [a, b]$, and $\varepsilon > 0$, then $\exists\ g_\xi \in \mathcal{B}$ such that

$$g_\xi(\xi) = f(\xi) \quad \text{and} \quad g_\xi(x) < f(x) + \varepsilon \quad \text{for all } x \in [a, b].$$

For each $\xi, \eta \in [a, b]$ with $\xi \neq \eta$, let $g_{\xi\eta} \in \mathcal{A}$ denote a function satisfying **Step 4** above, that is $g_{\xi\eta}(\xi) = f(\xi), g_{\xi\eta}(\eta) = f(\eta)$. Fix $\xi \in [a, b]$. If $\eta \neq \xi$, define

$$U_\eta = \{x \in [a, b] : g_{\xi\eta}(x) < f(x) + \varepsilon\}.$$

Note that $\xi, \eta \in U_\eta$. Then U_η is the inverse image of $g_{\xi\eta} - f$ over the set $(-\infty, \varepsilon)$. By the continuity of $g_{\xi\eta} - f$, it follows that U_η is an open set in $[a, b]$. Since $\{U_\eta : \eta \in [a, b] \setminus \{\xi\}\}$ is an open cover of $[a, b]$ and $[a, b]$ is compact, there are $U_{\eta_1}, \ldots, U_{\eta_k}$ such that $[a, b] = U_{\eta_1} \cup \cdots \cup U_{\eta_k}$. Taking $g_\xi = \min\{g_{\xi\eta_1}, \ldots, g_{\xi\eta_k}\}$, then $g_\xi \in \mathcal{B}$ by **Step 3** and if $x \in U_{\eta_j}$ then $g_\xi(x) \leqslant g_{\xi\eta_j}(x) < f(x) + \varepsilon$, hence g_ξ has the desired property.

Step 6. If $f \in C([a, b])$ then $\exists\ g \in \mathcal{B}$ such that

$$f(x) - \varepsilon < g(x) < f(x) + \varepsilon \quad \text{for all } x \in [a, b].$$

For each $\xi \in [a, b]$, let g_ξ be a in **Step 5.** By the continuity of g_ξ and f, then

$$V_\xi = \{x \in [a, b] : g_\xi(x) > f(x) - \varepsilon\}$$

is an open set in $[a, b]$. Also, observe that $\xi \in V_\xi$. So $\{V_\xi : \xi \in [a, b]\}$ is an open cover of $[a, b]$, hence by the compactness of $[a, b]$, $\exists\ V_{\xi_1}, \ldots, V_{\xi_n}$ such that $[a, b] = V_{\xi_1} \cup \cdots \cup V_{\xi_n}$. Taking $g = \max\{g_{\xi_1}, \ldots, g_{\xi_n}\}$ it follows that $g \in \mathcal{B}$ by **Step 3**. Furthermore, if $x \in V_{\xi_j}$ then

$$g(x) \geqslant g_{\xi_j}(x) > f(x) - \varepsilon.$$

This implies $g(x) > f(x) - \varepsilon$ for $x \in [a, b]$.

To finish the proof of this theorem, let $\varepsilon > 0$ and $f \in C([a, b])$. By **Step 6** we can take $g \in \mathcal{B}$ such that $\|f - g\|_\infty \leqslant \varepsilon/2$ and by the definition of $\mathcal{B}$, $\exists\ h \in \mathcal{A}$ such that $\|g - h\|_\infty \leqslant \varepsilon/2$. Thus

$$\|f - h\|_\infty \leqslant \varepsilon,$$

by the triangle inequality. Thus $f \in \mathcal{B}$. □

Example 7.25. Let $\mathcal{A}$ be a subalgebra of $C([a,b])$. Assume $\mathcal{A}$ separates points in $[a,b]$ and it contains the constant functions. For $x_0 \in [a,b]$, prove that

$$\mathcal{A}(x_0) = \{g \in \mathcal{A} : g(x_0) = 0\}$$

is a subalgebra of $C([a,b])$ and if $f \in C([a,b])$ satisfies $f(x_0) = 0$, then $\forall \varepsilon > 0$, $\exists g \in \mathcal{A}$ such that $\|f - g\|_\infty \leqslant \varepsilon$.

Solution. It is easy to see that $\mathcal{A}(x_0) \neq \emptyset$. In fact, for any $g \in \mathcal{A}$ we have $g - g(x_0) \in \mathcal{A}(x_0)$. If $f, g \in \mathcal{A}(x_0)$, $\alpha \in \mathbb{R}$ then $(f+g)(x_0) = 0, (\alpha f)(x_0) = 0$, and $(fg)(x_0) = 0$, so $\mathcal{A}(x_0)$ is a subalgebra of $C([a,b])$.

Now assume $f \in C([a,b])$ and $f(x_0) = 0$. Let $\varepsilon > 0$. By the Stone-Weierstrass theorem, $\exists\, h \in \mathcal{A}$ such that $\|f - h\|_\infty \leqslant \varepsilon/2$. Since $f(x_0) = 0$ we have

$$|h(x_0)| \leqslant \varepsilon/2.$$

We define $g = h - h(x_0)$. Then $g \in \mathcal{A}(x_0)$. By the triangle inequality, it follows that

$$|f(x) - g(x)| \leqslant |f(x) - h(x)| + |h(x) - g(x)| \leqslant \varepsilon.$$

Thus g has the desired property. ■