Bearing: A device that supports, guides, and reduces the friction of motion between fixed and moving machine parts.

7.2-7.3 Contact stresses

Contact stress causes deformation, plastic or elastic. The contact area will change depending on the magnitude of the contact stress. Therefore, it is very important to calculate the actual stress at the point of contact, the so-called contact stress.

By theory of elasticity, developed by Hertz, in 1881, when he was 23 years old!

Simplifying Assumptions to Hertz’s Theory
Hertz’s model of contact stress is based on the following simplifying assumptions [6]:

- the materials in contact are homogeneous and the yield stress is not exceeded,
- contact stress is caused by the load which is normal to the contact tangent plane which effectively means that there are no tangential forces acting between the solids,
- the contact area is very small compared with the dimensions of the contacting solids,
- the contacting solids are at rest and in equilibrium,
- the effect of surface roughness is negligible.
In a static contact, the interface is the principal plane (no shear stress). Maximum stress occurs at 45 degree to the normal stress, and it can cause deformation by movement of dislocation.

Below contacting surface, \( \sigma_x, \sigma_z \), and shear stress varies with depth. Variation depends on the magnitude of the load. However, generally max shear stress occurs at about 0.6a, where a is radius of contact area.

Contact stress under sliding. Sliding generates friction force which is tangential force at interface. Principal plan rotates by angle \( \phi \).

\[ \phi = \frac{1}{2} \cos^{-1}(\mu/q/k) \]

Distribution of shear stress, in sliding contact, under contacting surface. The calculation is for friction coeff. of 0.2. To be noted is the location of maximum shear stress moves closer to the surface, comparing to the static contact.
**Radius of curvature**

**Center of curvature**

**Convex**: Center of curvature is inside the material (positive curvature)

**Concave**: Center of curvature is outside the material (negative curvature)

Reduced radius of curvature:

\[
\frac{1}{R'} = \frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{R_{xx}} + \frac{1}{R_{yy}}
\]

\[R_{xx} = R_{yy} = \infty\]

Two elastic bodies with convex surface in contact.

\[\frac{1}{R} = \frac{1}{R_x} + \frac{1}{R_y} + \frac{1}{R_{xx}} + \frac{1}{R_{yy}}\]

or

\[\frac{1}{R} = \left(\frac{1}{R_x} + \frac{1}{R_y}\right) - \left(\frac{1}{R_{xx}} + \frac{1}{R_{yy}}\right)\]
Table 7.1: Formulæ for contact parameters between two spheres.

<table>
<thead>
<tr>
<th>Contact area dimensions</th>
<th>Maximum contact pressure $p_{\text{max}}$</th>
<th>Average contact pressure $p_{\text{ave}}$</th>
<th>Maximum deflection $\delta$</th>
<th>Maximum shear stress $\tau_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = \frac{3WR^<em>}{E^</em>}$</td>
<td>$p_{\text{max}} = \frac{3W}{2\pi a^3}$</td>
<td>$p_{\text{ave}} = \frac{W}{\pi a^3}$</td>
<td>$\delta = 1.0397 \left( \frac{W^{3/2}}{E^* R^*} \right)$</td>
<td>$\tau_{\text{max}} = \frac{1}{3} p_{\text{max}}$ at a depth of $z = 0.6358 a$</td>
</tr>
</tbody>
</table>

where:

- $a$ is the radius of the contact area [m];
- $W$ is the normal load [N];
- $p$ is the contact pressure (Hertzian stress) [Pa];
- $\delta$ is the total deflection at the centre of the contact (i.e., $\delta = \delta_c + \delta_s$; where $\delta_c$ and $\delta_s$ are the maximum deflections of body 'A' and 'B', respectively) [m];
- $t$ is the shear stress [Pa];
- $z$ is the depth under the surface where the maximum shear stress acts [m];
- $E'$ is the reduced Young's modulus [Pa];
- $R'$ is the reduced radius of curvature [m].

\[
\frac{1}{R'} = \frac{1}{R_A} + \frac{1}{R_B} \quad \text{and} \quad \frac{1}{R'} = \frac{1}{R_A} + \frac{1}{R_B} = 2 \left( \frac{1}{R_A} + \frac{1}{R_B} \right)
\]

### EXAMPLE

Find the contact parameters for two steel balls. The normal force is $W = 5$ [N], the radii of the balls are $R_A = 10 \times 10^{-3}$ [m] and $R_B = 15 \times 10^{-3}$ [m]. The Young's modulus for both balls is $E = 2.1 \times 10^{11}$ [Pa] and the Poisson's ratio of steel is $\nu = 0.3$.

#### Reduced Radius of Curvature
Since $R_A = R_B = R = 10 \times 10^{-3}$ [m] and $R_B = R_A = R = 15 \times 10^{-3}$ [m] the reduced radius of curvature in the 'x' and 'y' directions are:

\[
\frac{1}{R_x} = \frac{1}{10 \times 10^{-3}} + \frac{1}{15 \times 10^{-3}} = 166.67 \implies R_x = 6 \times 10^{-3} [m]
\]

\[
\frac{1}{R_y} = \frac{1}{10 \times 10^{-3}} + \frac{1}{15 \times 10^{-3}} = 166.67 \implies R_y = 6 \times 10^{-3} [m]
\]

Note that $1/R_A = 1/R_B$, i.e., condition (7.3) is satisfied (circular contact), and the reduced radius of curvature is:

\[
\frac{1}{R'} = \frac{1}{R_A} + \frac{1}{R_B} = 166.67 - 166.67 = 333.34 \implies R' = 3 \times 10^{-3} [m]
\]

- Reduced Young's Modulus

\[
\frac{1}{E'} = \frac{1}{E} \left( \frac{1 - \nu_A^2}{R_A} + \frac{1 - \nu_B^2}{R_B} \right) = \frac{1}{2.1 \times 10^{11}} \left( \frac{1 - 0.3^2}{10^{-3}} + \frac{1 - 0.3^2}{15^{-3}} \right) = 2.308 \times 10^{10} [Pa]
\]

- Contact Area Dimensions

\[
a = \frac{3WR^*}{E^*} = \frac{3 \times 5 \times (3 \times 10^{-3})}{2.308 \times 10^{10}} = 5.799 \times 10^{-3} [m]
\]

- Maximum and Average Contact Pressures

\[
p_{\text{max}} = \frac{3W}{2\pi a^2} = \frac{3 \times 5}{2\pi (5.799 \times 10^{-3})^2} = 709.9 [kPa]
\]

\[
p_{\text{ave}} = \frac{W}{\pi a^2} = \frac{5}{\pi (5.799 \times 10^{-3})^2} = 472.3 [kPa]
\]

- Maximum Deflection

\[
\delta = 1.0397 \left( \frac{W^{3/2}}{E^* R^*} \right) = 1.0397 \left( \frac{5^{3/2}}{2.308 \times 10^{10} \times 3 \times 10^{-3}} \right) = 5.6 \times 10^{-2} [m]
\]

- Maximum Shear Stress

\[
\tau_{\text{max}} = \frac{1}{3} p_{\text{max}} = \frac{1}{3} 709.9 = 236.6 [kPa]
\]

- Depth at which Maximum Shear Stress Occurs

\[
z = 0.6358a = 0.6358 \times (5.799 \times 10^{-3}) = 3.7 \times 10^{-3} [m]
\]
EXAMPLE
Find the contact parameters for a steel ball on a flat steel plate. The normal force is \( W = 5 \) [N], the radius of the ball is \( R_a = 10 \times 10^{-3} \) [m], the Young's modulus for ball and plate is \( E = 2.1 \times 10^7 \) [Pa] and the Poisson's ratio is \( \nu = 0.3 \).

Reduced Radii of Curvature
Since the radii of the ball and the plate are \( R_a = R_p = 10 \times 10^{-3} \) [m] and \( R_a = R_p = \infty \) [m], respectively, the reduced radii of curvature in 'x' and 'y' directions are:

\[
\frac{1}{R_x} = \frac{1}{R_a} + \frac{1}{R_p} = \frac{1}{10 \times 10^{-3}} + \frac{1}{\infty} = 100
\]

\[
\Rightarrow R_x = 0.01 \text{ [m]}
\]

\[
\frac{1}{R_y} = \frac{1}{R_a} + \frac{1}{R_p} = \frac{1}{10 \times 10^{-3}} + \frac{1}{\infty} = 100
\]

\[
\Rightarrow R_y = 0.01 \text{ [m]}
\]

Condition (7.3), i.e., \( 1/R_x = 1/R_y \) is satisfied (circular contact), and the reduced radius of curvature is:

\[
\frac{1}{R'} = \frac{1}{R_x} + \frac{1}{R_y} = 100 + 100 = 200
\]

\[
\Rightarrow R' = 5 \times 10^{-3} \text{ [m]}
\]

Reduced Young's Modulus

\[ E' = 2.308 \times 10^7 \text{ [Pa] } \]
**Table 7.2** Formulae for contact parameters between two parallel cylinders.

<table>
<thead>
<tr>
<th>Contact dimensions</th>
<th>Maximum contact pressure</th>
<th>Average contact pressure</th>
<th>Maximum deflection</th>
<th>Maximum shear stress</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_{max} = \frac{W}{\pi b l}$</td>
<td>$p_{average} = \frac{W}{4b l}$</td>
<td>$\delta = \frac{0.319}{\sqrt{\frac{W}{E l}}}$</td>
<td>$\tau_{max} = 0.304 p_{max}$ at a depth of $z = 0.786 b$</td>
</tr>
</tbody>
</table>

**EXAMPLE**

Find the contact parameters for two parallel steel rollers. The normal force is $W = 5$ [N], radii of the rollers are $R_x = 10 \times 10^3$ [m] and $R_y = 15 \times 10^3$ [m], Young's modulus for both rollers is $E = 2.1 \times 10^{11}$ [Pa] and the Poisson's ratio is $\nu = 0.3$. The length of both rollers is $2l = 10 \times 10^4$ [m].

**Reduced Radius of Curvature**

Since the radii of the cylinders are $R_x = R_y = 10 \times 10^3$ [m], $R_{x+y} = \infty$ and $R_{x-y} = R_{y-x} = 15 \times 10^3$ [m], the reduced radius of curvature in the 'x' and 'y' directions are:

$$\frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{10 \times 10^3} + \frac{1}{15 \times 10^3} = 166.67$$

$$\Rightarrow R = 6 \times 10^3$$ [m]

$$\frac{1}{R_{x+y}} = \frac{1}{R_{x-y}} = 0$$

$$\Rightarrow R = \infty$$ [m]

Since $1/R_x > 1/R_y$ condition (7.3) is satisfied and the reduced radius of curvature is:

$$\frac{1}{R_x} + \frac{1}{R_y} = 166.67$$

$$\Rightarrow R = 6 \times 10^3$$ [m]

**Reduced Young's Modulus**

$$E' = 2.308 \times 10^{11}$$ [Pa]

**Contact Area Dimensions**

$$b = \frac{4WR}{\pi E} = \frac{4 \times 5 \times (6 \times 10^{-2})}{\pi \times (5 \times 10^{-3}) \times (2.308 \times 10^{11})} = 5.75 \times 10^{-4}$$ [m]

**Maximum and Average Contact Pressures**

$$p_{max} = \frac{W}{\pi b l} = \frac{5}{\pi \times (5.75 \times 10^{-4}) \times (5 \times 10^{-3})} = 55.4$$ [MPa]

$$p_{average} = \frac{W}{4b l} = \frac{5}{4 \times (5.75 \times 10^{-4}) \times (5 \times 10^{-3})} = 43.5$$ [MPa]

**Maximum Deflection**

$$\delta = \frac{0.319}{\sqrt{\frac{W}{E l}}} \left[ \frac{5}{(2.308 \times 10^{11}) \times (5 \times 10^{-3})} \right] \left[ \frac{2}{3} + \ln \left( \frac{4 \times (10 \times 10^3) \times (15 \times 10^3)}{(5.75 \times 10^{-4})^2} \right) \right]$$

$$= 0.319 \left[ \frac{5}{2.308 \times 10^{11} \times 5 \times 10^{-3}} \right] \left[ \frac{2}{3} + \ln \left( \frac{4 \times 10 \times 15}{(5.75 \times 10^{-4})^2} \right) \right]$$

$$= 2.40 \times 10^{-3}$$ [m]

**Maximum Shear Stress**

$$\tau_{max} = 0.304 p_{max} = 0.304 \times 55.4 = 16.8$$ [MPa]

**Depth at which Maximum Shear Stress Occurs**

$$z = 0.786 b = 0.786 \times (5.75 \times 10^{-4}) = 4.5 \times 10^{-4}$$ [m]
EXAMPLE

Find the contact parameters for two steel wires of the same diameter crossed at 90°. This configuration is often used in fretting wear studies. The normal force is $N = 5$ [N], radii of the wires are $R_a = R_b = 1.5 \times 10^{-3}$ [m], the Young’s modulus for both wires is $E = 2.1 \times 10^11$ [Pa] and the Poisson’s ratio is $v = 0.3$.

- Reduced Radius of Curvature

Since the radii of the wires are $R_a = R_b = 1.5 \times 10^{-3}$ [m], and $R_a = R_b = 1.5 \times 10^{-3}$ [m], respectively, the reduced radius of curvature in the $x'$ and $y'$ directions are:

$$\frac{1}{R_a} = \frac{1}{R_{a*}} + \frac{1}{1.5 \times 10^{-3}} \Rightarrow R_a = 0.0015$$ [m]

$$\frac{1}{R_b} = \frac{1}{R_{b*}} + \frac{1}{1.5 \times 10^{-3}} \Rightarrow R_b = 0.0015$$ [m]

Since $1/R_a = 1/R_b$ condition (7.3) is satisfied and the reduced radius of curvature is:

$$\frac{1}{R} = \frac{1}{R_a} + \frac{1}{R_b} \Rightarrow R' = 7.5 \times 10^{-4}$$ [m]

- Reduced Young’s Modulus

$$E' = 2.308 \times 10^{11}\text{[Pa]}$$

Contact Area Dimensions

$$a = \left(\frac{3W}{E}\right)^{\frac{1}{3}} = 3.65 \times 10^{-4}$$ [m]

Maximum and Average Contact Pressures

$$p_{max} = \frac{3W}{2 \pi a^2} = \frac{3 \times 5}{2 \times (3.65 \times 10^{-4})^2} \Rightarrow p_{max} = 1791.9\text{[MPa]}

$$p_{avg} = \frac{W}{\pi a^2} = \frac{5}{(3.65 \times 10^{-4})^2} \Rightarrow p_{avg} = 1194.6\text{[MPa]}

Maximum Deflection

$$\delta = 1.6397 \left(\frac{W^2}{E \pi a^4}\right)^{\frac{1}{3}} \Rightarrow \delta = 8.9 \times 10^{-4}$$ [m]

Maximum Shear Stress

$$\tau_{max} = \frac{1}{3} P_{max} \Rightarrow \tau_{max} = 597.3\text{[MPa]}

Depth at which Maximum Shear Stress Occurs

$$z = 0.638 \times 9.638 \times (3.65 \times 10^{-4}) \Rightarrow z = 2.3 \times 10^{-4}$$ [m]

TABLE 7.4 Approximate formulae for contact parameters between two elastic bodies [7].

<table>
<thead>
<tr>
<th>Contact area dimensions</th>
<th>Maximum contact pressure</th>
<th>Maximum deflection</th>
<th>Simplified elliptical integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = \frac{6E W}{(\pi E')^2}$</td>
<td>$p_{max} = \frac{3W}{2 \pi a^2}$</td>
<td>$\delta = \left(\frac{4.5}{e^2} \frac{W}{E} \frac{1}{\pi a^2}\right)^{\frac{1}{3}}$</td>
<td>$\bar{e} = 1.0003 + 0.5968 \frac{R_a}{R_b}$</td>
</tr>
<tr>
<td>$b = \frac{6E W}{(\pi E')^2}$</td>
<td>Average contact pressure</td>
<td>$\bar{\delta} = 1.2577 + 0.6023 \ln \left(\frac{R_a}{R_b}\right)$</td>
<td>Ellipticity parameter</td>
</tr>
<tr>
<td>$p_{avg} = \frac{W}{\pi a b}$</td>
<td></td>
<td></td>
<td>$k = 1.0339 \frac{R_b}{R_a}$</td>
</tr>
</tbody>
</table>

| $\varepsilon = \frac{4.5}{e^2} \frac{W}{E} \frac{1}{\pi a^2}$ | | | |
EXAMPLE

Find the contact parameters for a steel ball in contact with a groove on the inside of a steel ring (as shown in Figure 7.7). The normal force is \( W = 50 [N] \), radius of the ball is \( R_b = 15 \times 10^{-3} [m] \), the radius of the groove is \( R_g = 30 \times 10^{-3} [m] \) and the radius of the ring is \( R_s = 60 \times 10^{-3} [m] \). The Young's modulus for both ball and ring is \( E = 2.1 \times 10^11 [Pa] \) and the Poisson's ratio is \( \nu = 0.3 \).

- **Reduced Radius of Curvature**

Since the radii of the ball and the grooved ring are \( R_b = 15 \times 10^{-3} [m] \), \( R_g = 30 \times 10^{-3} [m] \) and \( R_s = 60 \times 10^{-3} [m] \) (concave surface), \( R_b = 60 \times 10^{-3} [m] \) (concave surface), respectively, the reduced radii of curvature in the 'x' and 'y' directions are:

\[
\frac{1}{R_x} = \frac{1}{R_b} + \frac{1}{R_s} = \frac{1}{15 \times 10^{-3}} + \frac{1}{60 \times 10^{-3}} = 33.33 \quad \Rightarrow \quad R_x = 0.03 [m]
\]

\[
\frac{1}{R_y} = \frac{1}{R_b} + \frac{1}{R_s} = \frac{1}{15 \times 10^{-3}} + \frac{1}{60 \times 10^{-3}} = 50 \quad \Rightarrow \quad R_y = 0.02 [m]
\]

Since \( 1/R_x \), \( R_s \) condition (7.3) is not satisfied. According to the convention it is necessary to transpose the directions of the coordinates, so 'R_x' and 'R_y' become:

\( R_x = 0.02 [m] \) and \( R_y = 0.03 [m] \)

...and the reduced radius of curvature is:

\[
\frac{1}{R'} = \frac{1}{R_x} + \frac{1}{R_y} = \frac{50.0 + 33.33}{83.33} = 0.612 [m]
\]

**Reduced Young's Modulus**

\[
E' = 2.36 \times 10^11 [Pa]
\]

**Simplified Elliptical Integrals**

\[\frac{\xi}{\beta} = \frac{0.5968 R_s}{R_s} = 1 \]

\[\frac{\xi}{\beta} = 1.5277 + 0.6023 \ln \frac{R_s}{R_s} = 1.5277 + 0.6023 \ln \frac{0.03}{0.02} = 1.3982 \]

**Contact Area Dimensions**

\[a = \frac{6 \pi^3 \beta W}{\pi^3 E'} = \frac{6 \times 1.338 \times 0.03}{2.36 \times 10^11} = 2.32 \times 10^{-6} [m^2]\]

\[b = \frac{\pi \beta W R_s^3}{\pi \beta E'} = \frac{6 \times 1.338 \times 0.03}{2.36 \times 10^11} = 1.73 \times 10^{-4} [m] \]

**Maximum and Average Contact Pressures**

\[p_{\text{max}} = \frac{3W}{2\pi ab} = \frac{3 \times 50}{2 \pi (2.32 \times 10^{-4}) \times (1.73 \times 10^{-4})} = 594.8 [MPa]\]

\[p_{\text{ave}} = \frac{W}{\pi ab} = \frac{50}{\pi (2.32 \times 10^{-4}) \times (1.73 \times 10^{-4})} = 396.5 [MPa]\]

**Maximum Deflection**

\[\delta = 6 \left[ \frac{4.5}{E'} \right] \left[ \frac{W}{3 \pi ab} \right]^{3/2} \]

\[= 1.7719 \left[ \frac{4.5}{1.338 \times 0.012} \right]^{3/2} = 1.6 \times 10^{-4} [m]\]