Radon-Nikodym theorem

Theorem 1. *Let ν and μ be σ-finite positive measures defined on a measurable space $(X, \mathfrak{M})$ such that $\nu \ll \mu$. Then there exists a function $f\colon X \to [0, \infty]$ such that*

$$\nu(E) = \int_E f \, d\mu$$

for any $E \in \mathfrak{M}$. The function f is unique up to a set of measure zero.

Proof. First, we prove the theorem under the assumption that μ and ν are finite measures.

Let $\Lambda\colon L^2(X, \nu + \mu) \to \mathbb{C}$ be defined by

$$\Lambda(g) = \int_X g \, d\nu.$$

By Hölder's inequality,

$$|\Lambda(g)| \leq \left\{ \int_X 1^2 \, d\nu \right\}^{1/2} \left\{ \int_X |g|^2 \, d(\nu + \mu) \right\}^{1/2} = \nu(X)^{1/2} \|g\|_{L^2(\nu+\mu)}.$$

Hence, Λ is a bounded linear functional on the Hilbert space $L^2(X, \nu + \mu)$. By Riesz's theorem, there exists a unique $h \in L^2(X, \nu + \mu)$ such that

$$\Lambda(g) = \int_X g \, d\nu = \int_X hg \, d(\nu + \mu) \tag{1}$$

for any $g \in L^2(X, \nu + \mu)$. Putting $g = \chi_E$ in (1), we have the following equation

$$\nu(E) = \int_E h \, d(\nu + \mu) \tag{2}$$

for any measurable set E. Now let $E = \{x \in X \mid \operatorname{Im} h(x) > 0\}$. Then (2) becomes

$$\nu(E) = \int_E (\operatorname{Re} h + i \operatorname{Im} h) \, d(\nu + \mu).$$

This implies that $\int_E (\operatorname{Im} h) \, d(\nu + \mu) = 0$, and that $(\nu + \mu)(E) = 0$. Similarly, if $E = \{x \in X \mid \operatorname{Im} h(x) < 0\}$, then $(\nu + \mu)(E) = 0$. Hence, $h(x) \in \mathbb{R}$ $(\nu + \mu)$-a.e. By modifying h on a set of measure zero, we may assume that h is a real-valued function. Now, let $E = \{x \in X \mid h(x) < 0\}$ in (2). Since $h(x) < 0$ on E, it follows that $(\mu + \nu)(E) = 0$ or else the right side of this equation would be negative. Hence $h(x) \geq 0$ $(\nu + \mu)$-a.e. Again, by modifying h on a set of measure zero, we may assume that $h \geq 0$ everywhere. Now, let $E = \{x \in X \mid h(x) \geq 1\}$. Then

$$\nu(E) = \int_E h \, d(\nu + \mu) \geq \int_E d(\nu + \mu) = (\nu + \mu)(E).$$

Hence $\nu(E) = 0$. Since $\nu \ll \mu$, it follows that $\nu(E) = 0$ and that $(\nu + \mu)(E) = 0$. Thus $h(x) < 1$ $(\nu + \mu)$-a.e. Again we can assume that $0 \le h < 1$ everywhere.

From formula (1), for any $g \in L^2(X, \nu + \mu)$ and any $E \in \mathfrak{M}$, we have

$$\int_E g(1-h)\, d\nu = \int_E hg\, d\mu. \tag{3}$$

Putting $g = \sum_{k=0}^{n} h^k$ in (3), we have

$$\int_E (1 - h^{n+1})\, d\nu \;=\; \int_E \sum_{k=1}^{n+1} h^k\, d\mu.$$

Applying Monotone Convergence Theorem to the above equation, it follows that

$$\nu(E) \;=\; \int_E \sum_{k=1}^{\infty} h^k\, d\mu.$$

Let $f = \sum_{k=1}^{\infty} h^k = \frac{h}{1-h}$. Then $f\colon X \to [0, \infty)$ is measurable and

$$\nu(E) = \int_E f\, d\mu \qquad \text{for any } E \in \mathfrak{M}.$$

Since ν is a finite measure and f is nonnegative, it follows that $f \in L^1(X, \mu)$. The uniqueness of f (up to a set of measure zero) follows from the following statement:

If $f \in L^1(\mu)$ and $\int_E f\, d\mu = 0$ for any $E \in \mathfrak{M}$, then $f = 0$ a.e.

Now assume that μ and ν are σ-finite measures. Then X can be decomposed as a countable disjoint union of sets which are μ-finite and a countable disjoint union of sets which are ν-finite. By taking intersection of these sets, we may assume that $X = \bigcup_{n=1}^{\infty} A_n$, where A_n are pairwise disjoint and both $\mu(A_n)$ and $\nu(A_n)$ are finite. For each n, let

$$\mu_n(E) = \mu(E \cap A_n) \quad \text{and} \quad \nu_n(E) = \nu(E \cap A_n)$$

for any $E \in \mathfrak{M}$. Then ν_n and μ_n are finite measures. Moreover, $\nu \ll \mu$ implies $\nu_n \ll \mu_n$. Hence, for each $n \in \mathbb{N}$ there is a measurable function $f_n\colon X \to [0, \infty)$ such that

$$\nu_n(E) = \int_E f_n\, d\mu_n \qquad \text{for each } E \in \mathfrak{M}.$$

Note that $\mu_n(E) = \mu(E \cap A_n) = \int_E \chi_{A_n}\, d\mu$. Hence,

$$\begin{aligned}\nu(E) \;&=\; \sum_{n=1}^{\infty} \nu_n(E) = \sum_{n=1}^{\infty} \int_E f_n\, d\mu_n \\ &= \sum_{n=1}^{\infty} \int_E f_n \chi_{A_n}\, d\mu = \int_E \sum_{n=1}^{\infty} f_n \chi_{A_n}\, d\mu.\end{aligned}$$

Define $f = \sum_{n=1}^{\infty} f_n \chi_{A_n}$. Then f has the desired property.

Finally, we establish the uniqueness of function f (up to null sets). Suppose that $f,\, g\colon X \to [0,\infty)$ are measurable functions such that

$$\nu(E) = \int_E f\, d\mu = \int_E g\, d\mu$$

for every $E \in \mathfrak{M}$. Then the finite case shows that $f = g$ μ-a.e. on each A_n. Hence, $f = g$ μ-a.e. on X. □

Theorem 2 (Lebesgue Decomposition). *Let ν and μ be σ-finite measures on a measurable space $(X, \mathfrak{M})$. Then there exists a unique pair of measures ν_a and ν_s such that $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ and $\nu_s \perp \mu$.*

Proof. Let $\lambda = \mu + \nu$. Since μ and ν are σ-finite, so is λ. Moreover, $\mu \ll \lambda$ (and also $\nu \ll \lambda$). By Radon-Nikodym theorem, there is a function $f\colon X \to [0,\infty)$ such that

$$\mu(E) = \int_E f\, d\lambda \qquad \text{for any } E \in \mathfrak{M}.$$

Let $A = \{x \in X \mid f(x) > 0\}$ and $B = \{x \in X \mid f(x) = 0\}$. Then $A,\, B \in \mathfrak{M}$, $X = A \cup B$ and $A \cap B = \varnothing$. Define

$$\nu_a(E) = \nu(E \cap A) \quad \text{and} \quad \nu_s(E) = \nu(E \cap B)$$

for any $E \in \mathfrak{M}$. Clearly $\nu = \nu_a + \nu_s$. Furthermore, $\nu_s(A) = 0$ and $\mu(B) = 0$. Hence $\nu_s \perp \mu$. It remains to show that $\nu_a \ll \mu$. Let $E \in \mathfrak{M}$ be such that $\mu(E) = 0$. Then $\mu(E \cap A) = 0$. It follows that $\int_{E\cap A} f\, d\mu = 0$. Since $f(x) > 0$ on A, we must have $\lambda(E \cap A) = 0$. Hence, $\nu_a(E) = \nu(E \cap A) = 0$. Thus $\nu_a \ll \mu$.

We now establish the uniqueness of the pair (ν_a, ν_s). Suppose that $(\tilde{\nu}_a, \tilde{\nu}_s)$ is another pair such that

$$\nu = \tilde{\nu}_a + \tilde{\nu}_s, \quad \tilde{\nu}_a \ll \mu, \quad \tilde{\nu}_s \perp \mu.$$

First, we assume that all the measures involved are finite measures. In this case we have $\tilde{\nu}_a - \nu_a = \nu_s - \tilde{\nu}_s$. Moreover, $\tilde{\nu}_a - \nu_a \ll \mu$ and $\nu_s - \tilde{\nu}_s \perp \mu$. This implies that $\tilde{\nu}_a - \nu_a = \nu_s - \tilde{\nu}_s = 0$ and that $\tilde{\nu}_a = \nu_a$ and $\nu_s = \tilde{\nu}_s$. Here we use the fact that if $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.

In the σ-finite case, assume that $X = \bigcup_{n=1}^{\infty} A_n$, where A_n are pairwise disjoint and both $\mu(A_n)$ and $\nu(A_n)$ are finite. Define for any $E \in \mathfrak{M}$,

$$\begin{aligned} \sigma_n(E) &= \nu_a(E \cap A_n), & \tilde{\sigma}_n(E) &= \tilde{\nu}_a(E \cap A_n), \\ \varrho_n(E) &= \nu_s(E \cap A_n), & \tilde{\varrho}_n(E) &= \tilde{\nu}_s(E \cap A_n). \end{aligned}$$

Then $\sigma_n, \tilde{\sigma}_n \ll \mu_n$ and $\varrho_n, \tilde{\varrho}_n \perp \mu_n$ for each n. By the uniqueness part of the finite case, we have $\sigma_n = \tilde{\sigma}_n$ and $\varrho_n = \tilde{\varrho}_n$ for each n. Hence, $\nu_a = \sum_n \sigma_n = \sum_n \tilde{\sigma}_n = \tilde{\nu}_a$ and $\nu_s = \sum_n \varrho_n = \sum_n \tilde{\varrho}_n = \tilde{\nu}_s$. □