Theorem 1. Let $\nu$ and $\mu$ be $\sigma$-finite positive measures defined on a measurable space $(X, \mathcal{M})$ such that $\nu \ll \mu$. Then there exists a function $f: X \to [0, \infty]$ such that

$$\nu(E) = \int_E f \, d\mu$$

for any $E \in \mathcal{M}$. The function $f$ is unique up to a set of measure zero.

Proof. First, we prove the theorem under the assumption that $\mu$ and $\nu$ are finite measures.

Let $\Lambda: L^2(X, \nu + \mu) \to \mathbb{C}$ be defined by

$$\Lambda(g) = \int_X g \, d\nu.$$ 

By Hölder’s inequality,

$$|\Lambda(g)| \leq \left\{ \int_X 1^2 \, d\nu \right\}^{1/2} \left\{ \int_X |g|^2 \, d(\nu + \mu) \right\}^{1/2} = \nu(X)^{1/2} \|g\|_{L^2(\nu + \mu)}.$$ 

Hence, $\Lambda$ is a bounded linear functional on the Hilbert space $L^2(X, \nu + \mu)$. By Riesz’s theorem, there exists a unique $h \in L^2(X, \nu + \mu)$ such that

$$\Lambda(g) = \int_X g \, d\nu = \int_X hg \, d(\nu + \mu) \quad (1)$$

for any $g \in L^2(X, \nu + \mu)$. Putting $g = \chi_E$ in (1), we have the following equation

$$\nu(E) = \int_E h \, d(\nu + \mu) \quad (2)$$

for any measurable set $E$. Now let $E = \{ x \in X \mid \text{Im} \, h(x) > 0 \}$. Then (2) becomes

$$\nu(E) = \int_E (\text{Re} \, h + i \, \text{Im} \, h) \, d(\nu + \mu).$$

This implies that $\int_E (\text{Im} \, h) \, d(\nu + \mu) = 0$, and that $(\nu + \mu)(E) = 0$. Similarly, if $E = \{ x \in X \mid \text{Im} \, h(x) < 0 \}$, then $(\nu + \mu)(E) = 0$. Hence, $h(x) \in \mathbb{R} \, (\nu + \mu)$-a.e.

By modifying $h$ on a set of measure zero, we may assume that $h$ is a real-valued function. Now, let $E = \{ x \in X \mid h(x) < 0 \}$ in (2). Since $h(x) < 0$ on $E$, it follows that $(\mu + \nu)(E) = 0$ or else the right side of this equation would be negative. Hence $h(x) \geq 0 \, (\nu + \mu)$-a.e. Again, by modifying $h$ on a set of measure zero, we may assume that $h \geq 0$ everywhere. Now, let $E = \{ x \in X \mid h(x) \geq 1 \}$. Then

$$\nu(E) = \int_E h \, d(\nu + \mu) \geq \int_E d(\nu + \mu) = (\nu + \mu)(E).$$

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Hence \( \nu(E) = 0 \). Since \( \nu \ll \mu \), it follows that \( \nu(E) = 0 \) and that \( (\nu + \mu)(E) = 0 \). Thus \( h(x) < 1 \) \( (\nu + \mu) \)-a.e. Again we can assume that \( 0 \leq h < 1 \) everywhere.

From formula (1), for any \( g \in L^2(X, \nu + \mu) \) and any \( E \in \mathcal{M} \), we have

\[
\int_E g(1 - h) \, d\nu = \int_E h g \, d\mu. \tag{3}
\]

Putting \( g = \sum_{k=0}^{n} h^k \) in (3), we have

\[
\int_E (1 - h^{n+1}) \, d\nu = \int_E \sum_{k=1}^{n+1} h^k \, d\mu.
\]

Applying Monotone Convergence Theorem to the above equation, it follows that

\[
\nu(E) = \int_E \sum_{k=1}^{\infty} h^k \, d\mu.
\]

Let \( f = \sum_{k=1}^{\infty} h^k = \frac{h}{1 - h} \). Then \( f : X \to [0, \infty) \) is measurable and

\[
\nu(E) = \int_E f \, d\mu \quad \text{for any } E \in \mathcal{M}.
\]

Since \( \nu \) is a finite measure and \( f \) is nonnegative, it follows that \( f \in L^1(X, \mu) \). The uniqueness of \( f \) (up to a set of measure zero) follows from the following statement:

If \( f \in L^1(\mu) \) and \( \int_E f \, d\mu = 0 \) for any \( E \in \mathcal{M} \), then \( f = 0 \) a.e.

Now assume that \( \mu \) and \( \nu \) are \( \sigma \)-finite measures. Then \( X \) can be decomposed as a countable disjoint union of sets which are \( \mu \)-finite and a countable disjoint union of sets which are \( \nu \)-finite. By taking intersection of these sets, we may assume that \( X = \bigcup_{n=1}^{\infty} A_n \), where \( A_n \) are pairwise disjoint and both \( \mu(A_n) \) and \( \nu(A_n) \) are finite. For each \( n \), let

\[
\mu_n(E) = \mu(E \cap A_n) \quad \text{and} \quad \nu_n(E) = \nu(E \cap A_n)
\]

for any \( E \in \mathcal{M} \). Then \( \nu_n \) and \( \mu_n \) are finite measures. Moreover, \( \nu \ll \mu \) implies \( \nu_n \ll \mu_n \). Hence, for each \( n \in \mathbb{N} \) there is a measurable function \( f_n : X \to [0, \infty) \) such that

\[
\nu_n(E) = \int_E f_n \, d\mu_n \quad \text{for each } E \in \mathcal{M}.
\]

Note that \( \mu_n(E) = \mu(E \cap A_n) = \int_E \chi_{A_n} \, d\mu \). Hence,

\[
\nu(E) = \sum_{n=1}^{\infty} \nu_n(E) = \sum_{n=1}^{\infty} \int_E f_n \, d\mu_n
\]

\[
= \sum_{n=1}^{\infty} \int_E f_n \chi_{A_n} \, d\mu = \int_E \sum_{n=1}^{\infty} f_n \chi_{A_n} \, d\mu.
\]

Define \( f = \sum_{n=1}^{\infty} f_n \chi_{A_n} \). Then \( f \) has the desired property.
Finally, we establish the uniqueness of function $f$ (up to null sets). Suppose that $f, g : X \to [0, \infty)$ are measurable functions such that

$$
\nu(E) = \int_E f \, d\mu = \int_E g \, d\mu
$$

for every $E \in \mathcal{M}$. Then the finite case shows that $f = g$ $\mu$-a.e. on each $A_n$. Hence, $f = g$ $\mu$-a.e. on $X$.

\[ \square \]

**Theorem 2 (Lebesgue Decomposition).** Let $\nu$ and $\mu$ be $\sigma$-finite measures on a measurable space $(X, \mathcal{M})$. Then there exists a unique pair of measures $\nu_a$ and $\nu_s$ such that $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

**Proof.** Let $\lambda = \mu + \nu$. Since $\mu$ and $\nu$ are $\sigma$-finite, so is $\lambda$. Moreover, $\mu \ll \lambda$ (and also $\nu \ll \lambda$). By Radon-Nikodym theorem, there is a function $f : X \to [0, \infty)$ such that

$$
\mu(E) = \int_E f \, d\lambda \quad \text{for any } E \in \mathcal{M}.
$$

Let $A = \{ x \in X \mid f(x) > 0 \}$ and $B = \{ x \in X \mid f(x) = 0 \}$. Then $A, B \in \mathcal{M}$, $X = A \cup B$ and $A \cap B = \emptyset$. Define

$$
\nu_a(E) = \nu(E \cap A) \quad \text{and} \quad \nu_s(E) = \nu(E \cap B)
$$

for any $E \in \mathcal{M}$. Clearly $\nu = \nu_a + \nu_s$. Furthermore, $\nu_s(A) = 0$ and $\mu(B) = 0$. Hence $\nu_s \perp \mu$. It remains to show that $\nu_a \ll \mu$. Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$. Then $\mu(E \cap A) = 0$. It follows that $\int_{E \cap A} f \, d\mu = 0$. Since $f(x) > 0$ on $A$, we must have $\lambda(E \cap A) = 0$. Hence, $\nu_a(E) = \nu(E \cap A) = 0$. Thus $\nu_a \ll \mu$.

We now establish the uniqueness of the pair $(\nu_a, \nu_s)$. Suppose that $(\tilde{\nu}_a, \tilde{\nu}_s)$ is another pair such that

$$
\nu = \tilde{\nu}_a + \tilde{\nu}_s, \quad \tilde{\nu}_a \ll \mu, \quad \tilde{\nu}_s \perp \mu.
$$

First, we assume that all the measures involved are finite measures. In this case we have $\tilde{\nu}_a - \nu_a = \nu_s - \tilde{\nu}_s$. Moreover, $\tilde{\nu}_a - \nu_a \ll \mu$ and $\nu_s - \tilde{\nu}_s \perp \mu$. This implies that $\tilde{\nu}_a - \nu_a = \nu_s - \tilde{\nu}_s = 0$ and that $\tilde{\nu}_a = \nu_a$ and $\nu_s = \tilde{\nu}_s$. Here we use the fact that if $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.

In the $\sigma$-finite case, assume that $X = \bigcup_{n=1}^{\infty} A_n$, where $A_n$ are pairwise disjoint and both $\mu(A_n)$ and $\nu(A_n)$ are finite. Define for any $E \in \mathcal{M}$,

$$
\sigma_n(E) = \nu_a(E \cap A_n), \quad \tilde{\sigma}_n(E) = \tilde{\nu}_a(E \cap A_n),
$$

$$
\varrho_n(E) = \nu_s(E \cap A_n), \quad \tilde{\varrho}_n(E) = \tilde{\nu}_s(E \cap A_n).
$$

Then $\sigma_n, \tilde{\sigma}_n \ll \mu_n$ and $\varrho_n, \tilde{\varrho}_n \perp \mu_n$ for each $n$. By the uniqueness part of the finite case, we have $\sigma_n = \tilde{\sigma}_n$ and $\varrho_n = \tilde{\varrho}_n$ for each $n$. Hence, $\nu_a = \sum_n \sigma_n = \sum_n \tilde{\sigma}_n = \tilde{\nu}_a$ and $\nu_s = \sum_n \varrho_n = \sum_n \tilde{\varrho}_n = \tilde{\nu}_s$. \[ \square \]