Lemma9: If f is bounded and measurable on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt + F(a)$$

then $F'(x) = f(x)$ for almost all $x \in [a, b]$

Proof: By lemma 7, $\boxed{\text{if } f \text{ is integrable on } [a,b] \Rightarrow F \text{ is continuous function of bounded variation on } [a,b]}$
F is of bounded variation over $[a, b]$ and so By Corollary 6 $\boxed{\text{if } f \in BV \Rightarrow f' \text{ exists a.e.}}$ $F'(x)$ exists a.e. in $[a, b]$

Since f is bounded. Let $|f| \leq K$. Then setting $f_n(x) = \dfrac{F(x+h) - F(x)}{h}$ with $h = \dfrac{1}{n}$,
$\boxed{\text{Note. } f_n(x) \to F'(x) \text{ a.e. because } \frac{1}{n} \to 0}$

we have $f_n(x) = \dfrac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt] = \dfrac{1}{h}\int_x^{x+h} f(t)dt$, and so $|f_n| \leq K$

Since $f_n(x) \to F'(x)$ a.e., and $|f_n| \leq K$, the DCT implies that, for all $c \in [a, b]$

$$\begin{aligned}
\int_a^c F'(x)dx = \lim_{n\to\infty} \int_a^c f_n(x)dx &= \lim_{h\to 0} \frac{1}{h} \int_a^c \Big(F(x+h) - F(x)\Big)dx \\
&= \lim_{h\to 0} \frac{1}{h}\Big[\int_c^{c+h} F(x)dx - \int_a^{a+h} F(x)dx\Big] \\
&= F(c) - F(a) \quad \boxed{\text{****Since } F \text{ is continuous}} \\
&= \int_a^c f(x)dx
\end{aligned}$$

Hence $\int_a^c (F'(x) - f(x))dx = 0, \forall c \in [a, b]$ By Lemma 8,

$\boxed{\text{if } \int_a^x f(t)dt = 0, \forall x \in [a,b] \Rightarrow f(t) = 0 \text{ a.e. in } [a,b]}$ we have $F'(x) = f(x)$ a.e. ✠

**** If f is continuous at a, then $\lim_{h\to 0}\frac{1}{h}\int_a^{a+h} f(x)dx = f(a)$
proof: Since f is continuous at a, we have $\forall\epsilon > 0 \exists\delta > 0, |f(t) - f(a)| < \epsilon$ where $|t-a| < \delta$
Consider where $h < \delta$ then $|\frac{\int_a^{a+h} f(t)dt}{h} - f(a)| = \frac{1}{h}|\int_a^{a+h} f(t)dt - \int_a^{a+h} f(a)dt|$
$= \frac{1}{h}|\int_a^{a+h}(f(t) - f(a))dt| \leq \frac{1}{h}\int_a^{a+h}|f(t) - f(a)|dt \leq \frac{1}{h}\epsilon h = \epsilon$
Hence $\lim_{h\to 0}\frac{1}{h}\int_a^{a+h} f(x)dx = f(a)$

Theorem 10: Let f be an integrable function on $[a,b]$, and suppose that

$$F(x) = F(a) + \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$ for almost all x in $[a,b]$.

Proof: First, we assume that $f \geq 0$. Let $f_n(x) = \min\{f(x), n\}$, then $f - f_n \geq 0$

and so $G_n(x) = \int_a^x (f(t) - f_n(t))dt$ is an increasing function.

By Theorem 3 $\boxed{f \text{ increasing} \Rightarrow f' \text{ exists a.e.}}$

$G_n'(x)$ exists a.e. and we have $G_n'(x) \geq 0$.

Now we have f_n bounded by Lemma 9, $\boxed{f \text{ bounded} \Rightarrow \frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ a.e.}}$

we have $\frac{d}{dx}\int_a^x f_n(t)dt = f_n(x)$ a.e.

Consider $G_n(x) = \int_a^x (f(t) - f_n(t))dt$ we have

$F'(x) = G_n'(x) + \frac{d}{dx}\int_a^x f_n(t)dt \geq f_n(x)$ a.e.

Since n is arbitrary, we have $F'(x) \geq f(x)$

Consequently, $\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a)$.

Thus by Theorem 3, $\boxed{f \text{ increasing} \Rightarrow \int_a^b f'(x)dx \leq f(b) - f(a)}$

we have $F(b) - F(a) \geq \int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a)$.

Thus $\int_a^b (F'(x) - f(x))dx = 0$. Since $F'(x) - f(x) \geq 0$ a.e., this implies that $F'(x) - f(x) = 0$ a.e. Hence $F'(x) = f(x)$ a.e.

Next, we assume that f be an integrable function. Since $f = f^+ - f^-$ where $f^+, f^- \geq 0$, we have $F(x) = F(a) + \int_a^x f(t)dt = F(a) + \int_a^x [f^+(t) - f^-(t)]dt$.

Hence $F'(x) = f^+(x) - f^-(x) = f(x)$ a.e. ✠