Definition 1. *Let $\mathcal{I}$ be a collection of intervals. Then we say that $\mathcal{I}$ covers a set E in the sense of Vitali, if for each $\epsilon > 0$ and any $x \in E$, there is an interval $I \in \mathcal{I}$ such that $x \in I$ and $l(I) < \epsilon$. The intervals may be open, closed or half-open, but we do not allow degenerate intervals consisting of only one point.*

Lemma 1. *(Vitali Covering Lemma)*
Let E be a set of finite outer measure and $\mathcal{I}$ a collection of intervals that cover E in the sense of Vitali. Then, given $\epsilon > 0$, there is a finite disjoint collection $\{I_1, \ldots, I_N\}$ of intervals in $\mathcal{I}$ such that

$$m^*\left[E \setminus \bigcup_{n=1}^{N} I_n\right] < \epsilon$$

Proof. It suffices to prove the lemma in the case that each interval in $\mathcal{I}$ is closed, for otherwise we replace each interval by its closure and observe that the set of end points of $I_1, \ldots, I_N$ has measure zero.
Let O be an open set of finite outer measure containing E.
Since $\mathcal{I}$ is a Vitali covering of E, we may assume WLOG that each I of $\mathcal{I}$ is contained in O.
We choose a sequence $\{I_n\}$ of disjoint intervals of $\mathcal{I}$ by induction as follows:
Let I_1 be any interval in $\mathcal{I}$ and suppose that $I_1, \ldots, I_n$ are disjoint intervals in $\mathcal{I}$.
If $E \subseteq \bigcup_{k=1}^{n} I_k$, then $m^*\left[E \setminus \bigcup_{k=1}^{n} I_k\right] = m^*\emptyset = 0 < \epsilon$.
If not, let k_n be the supremum of the lengths of the intervals of $\mathcal{I}$ that do not meet any of the intervals $I_1, \ldots, I_n$, that is,

$$\mathcal{I}_n = \{I \in \mathcal{I} \mid I \cap \bigcup_{k=1}^{n} I_k = \emptyset\} \text{ and } \ k_n = \sup\{l(I) \mid I \in \mathcal{I}_n\}$$

Since $E \nsubseteq \bigcup_{k=1}^{n} I_k$, there is $x \in E \setminus \bigcup_{k=1}^{n} I_k$.
Since $\bigcup_{k=1}^{n} I_k$ is closed set, $d(x, \bigcup_{k=1}^{n} I_k) > 0$.
Since $\mathcal{I}$ is a Vitali covering of E, there is an interval $I \in \mathcal{I}$ such that $x \in I$ and $l(I) < d(x, \bigcup_{k=1}^{n} I_k)$. Then $I \in \mathcal{I}_n$. Hence $\mathcal{I}_n$ is not empty and so $k_n > 0$.
Since each $I \in \mathcal{I}$ is contained in O, $k_n = \sup\{l(I) \mid I \in \mathcal{I}_n\} \leq m^*O < \infty$.
Thus $0 < k_n < \infty$.
Since $k_n = \sup\{l(I) \mid I \in \mathcal{I}_n\}$, we can choose $I_{n+1} \in \mathcal{I}_n$ such that $l(I_{n+1}) > \frac{1}{2}k_n$.
Continue this process.

So we have a sequence either $E \subseteq \bigcup_{k=1}^{n} I_k$ for some n, in which case the proof is

complete, or a sequence $\{I_n\}$ of disjoint intervals in $\mathcal{I}$ such that $l(I_{n+1}) > \dfrac{1}{2}k_n$ for all $n \geq 2$.
Since $\{I_n\}$ is disjoint,

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} m^*(I_n) = m^*(\bigcup_{n=1}^{\infty} I_n) \leq m^*O < \infty.$$

Hence $\lim_{n\to\infty} l(I_n) = 0$.
Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that $\displaystyle\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}$.
For each $k > N$, let J_k be the interval with the same center as I_k and five times the length of I_k.
Then

$$m^*(\bigcup_{n=N+1}^{\infty} J_n) \leq \sum_{n=N+1}^{\infty} m^*(J_n) = \sum_{n=N+1}^{\infty} l(J_n) = \sum_{n=N+1}^{\infty} 5l(I_n) = 5\sum_{n=N+1}^{\infty} l(I_n) < \epsilon.$$

If we can prove that $E \setminus \displaystyle\bigcup_{n=1}^{N} I_n \subseteq \bigcup_{n=N+1}^{\infty} J_n$, then

$$m^*\big[E \setminus \bigcup_{n=1}^{N} I_n\big] \leq m^*(\bigcup_{n=N+1}^{\infty} J_n) < \epsilon.$$

So we finish the proof.
Let $x \in E \setminus \displaystyle\bigcup_{n=1}^{N} I_n$.
Since $\mathcal{I}$ is a Vitali covering of E, there exists an interval I_x in $\mathcal{I}$ which contains x and whose length is so small that I_x does not meet any of the intervals $I_1, \ldots, I_N$.
Let $n \in \mathbb{N}$. Assume that $I_x \cap I_i = \emptyset$ for all $i \leq n$.
Then $I_x \in \mathcal{I}_n$ and so $l(I_x) \leq k_n$.
Since $0 < l(I_x) \leq k_n < 2l(I_{n+1})$ and $\lim_{n\to\infty} l(I_n) = 0$,
$0 < l(I_x) \leq \lim_{n\to\infty} 2l(I_{n+1}) = 0$, a contradiction.
Hence there is $n \in \mathbb{N}$ such that $I_x \cap I_i \neq \emptyset$ for some $i \leq n$.
Let m be the smallest integer such that $I_x \cap I_m \neq \emptyset$. Moreover $m > N$.
Then $I_x \in \mathcal{I}_{m-1}$ and so $l(I_x) \leq k_{m-1} < 2l(I_m)$.
Let c be the center of the interval I_m and $y \in I_x \cap I_m$. Compute

$$|x-c| \leq |x-y|+|y-c| \leq l(I_x)+\frac{1}{2}l(I_m) < 2l(I_m)+\frac{1}{2}l(I_m) = \frac{5}{2}l(I_m) = \frac{1}{2}l(J_m).$$

It follow that $x \in J_m$ for some $m > N$ and so $x \in \displaystyle\bigcup_{n=N+1}^{\infty} J_n$, as desired. $\square$