Theorem 1. *Let f be an increasing real-valued function on the interval $[a,b]$. Then f is differentiable almost everywhere. The derivate f' is measurable and*

$$\int_a^b f'(x)dx \leq f(b) - f(a)$$

Proof. To show that f is differentiable a.e.. It suffices to show that the sets where any two derivates are unequal have measure zero. We consider only the set E where $D^+f(x) > D_-f(x)$, the sets arising from ohter combinations of derivates being similarly handled. Note that

$$\begin{aligned} E &:= \{x | D^+f(x) > D_-f(x)\} \\ &= \bigcup_{u,v\in\mathbb{Q}} \{x|D^+f(x) > u > v > D_-f(x)\} \\ &=: \bigcup_{u,v\in\mathbb{Q}} E_{u,v} \end{aligned}$$

We will show that $m(E) = 0$ by showing that $m^*(E_{u,v}) = 0$ for all $u, v \in \mathbb{Q}$.
Let $s := m^*(E_{u,v})$ and $\epsilon > 0$.
Let O be and open set such that $E_{u,v} \subseteq O$ and $m(O) < s + \epsilon$.
For each $x \in E_{u,v}$, there is an arbitrarily small h such that

$$[x-h, x] \subseteq O \quad \text{and} \quad f(x) - f(x-h) < vh.$$

Note that $E_{u,v}$ is a set of finite outer measure and

$$\mathcal{I}_1 = \{[x-h, x] \mid x \in E_{u,v} \text{ and } h > 0\}$$

is a collection of interval covering $E_{u,v}$ in the sense of Vitali.
Then, by lemma1., there exists a finite disjoint collection $\{I_1, I_2, ..., I_N\}$ of $\mathcal{I}_1$ such that

$$m^*[E_{u,v} - \bigcup_{i=1}^N I_i] < \epsilon.$$

Let $A = E_{u,v} \cap (\bigcup_{i=1}^N int(I_i))$. Then

$$A \subseteq \bigcup_{i=1}^N int(I_i) \text{ and } A = E_{u,v} - (E_{u,v} - \bigcup_{i=1}^N int(I_i)).$$

Thus

$$\begin{aligned} m^*(A) &= m^*(E_{u,v}) - m^*(E_{u,v} - \bigcup_{i=1}^N int(I_i)) \\ &> s - \epsilon. \end{aligned}$$

Then, summing over these intervals, we have

$$\sum_{i=1}^{N}[f(x_i) - f(x_i - h_i)] < v\sum_{i=1}^{N} h_i$$
$$< vm(O)$$
$$< v(s + \epsilon). \quad (1)$$

For each $y \in A$, y is the left endpoint of arbitrarily small interval

$$(y, y + k) \subseteq I_n, \text{ for some } n \in \mathbb{N}, \text{ and } f(y + k) - f(y) > uk.$$

Note that A is a set of finite outer measure and

$$\mathcal{I}_2 = \{(y, y + k)|\ y \in A \text{ and } k > 0\}$$

is a collection of interval covering A. By lemma1., there is a finite collection $\{J_1, J_2, ..., J_M\}$ of $\mathcal{I}_2$ such that $\bigcup_{i=1}^{M} J_i$ contains a subset of A of outer measure greater than $s - 2\epsilon$. Then

$$\sum_{i=1}^{M}[f(y_i + k_i) - f(y_i)] > u\sum_{i=1}^{M} k_i$$
$$> u(s - 2\epsilon). \quad (2)$$

Each interval J_i is contained in some interval I_n. Summing over those i for which $J_i \subset I_n$, we have

$$\sum f(y_i + k_i) - f(y_i) \leq f(x_n) - f(x_n - h_n)$$

since f is increasing. Thus, by (1) and (2),

$$v(s + \epsilon) > \sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \geq \sum_{i=1}^{M} f(y_i + k_i) - f(y_i) > u(s - 2\epsilon).$$

Since ϵ is arbitrary, $vs \geq us$. But $u > v$, then s must be zero.

This shows that

$$g(x) = \lim_{h\to\infty} \frac{f(h + h) - f(x)}{h}$$

is defined a.e.. Next, we will show that g is finite. Let

$$g_n(x) = n[f(x + \frac{1}{n}) - f(x)] \geq 0$$

for all $n \in \mathbb{N}$ where we set $f(x) = f(b)$ for $x \geq b$.

Then $g_n(x) \to g(x)$ for almost all x.

Since f is increasing, f is measurable which implies g_n is measurable for all n.

Then g is measurable. By Fatou's lemma,

$$\begin{aligned}\int_a^b g = \int_a^b \lim_{n\to\infty} g_n &= \int_a^b \lim_{n\to\infty} \inf g_n \\ &\le \lim_{n\to\infty} \inf \int_a^b g_n \\ &= \lim_{n\to\infty} \inf n \int_a^b [f(x+\frac{1}{n}) - f(x)]dx \\ &= \lim_{n\to\infty} \inf[n \int_b^{b+\frac{1}{n}} f - n \int_a^{a+\frac{1}{n}} f] \\ &= \lim_{n\to\infty} \inf[f(b) - n \int_a^{a+\frac{1}{n}} f] \\ &\le \lim_{n\to\infty} [f(b) - f(a)] \\ &= f(b) - f(a).\end{aligned}$$

Since $g_n \ge 0$, so is g. Then g is integrable. This shows that g is finite a.e..

Thus f is differentiable a.e. and $g = f'$ a.e.

Hence f' is measurable. □