4. Absolute Continuity

A real-valued function f defined on $[a,b]$ is said to be *absolutely continuous* on $[a,b]$ if

$\forall \epsilon > 0\ \exists \delta > 0\ \forall n \in \mathbb{N}\ \forall \{(x_i, x_i^{'})\}_{i=1}^{n}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} |x_i^{'} - x_i| < \delta \Longrightarrow \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i)| < \epsilon$$

Note that (i) every absolutely continuous function is uniformly continuous,

(ii) every indefinite integral $F(x) = \displaystyle\int_a^x f(t)dt$ for all $x \in [a,b]$

where f is integrable on $[a,b]$ is absolutely continuous.

Proof. Let $\epsilon > 0$.

By Proposition 4.14; there is a $\delta > 0$ such that $\forall \{(x_i, x_i^{'})\}_{i=1}^{n}$ of

nonoverlapping intervals in $[a,b]$ with $m(\displaystyle\bigcup_{i=1}^{n}(x_i, x_i^{'})) = \sum_{i=1}^{n} |x_i^{'} - x_i| < \delta$

we have

$$\begin{aligned}
\sum_{i=1}^{n} |F(x_i^{'}) - F(x_i)| &= \sum_{i=1}^{n} \left| \int_{x_i}^{x_i^{'}} f(t)dt \right| \\
&\le \sum_{i=1}^{n} \int_{x_i}^{x_i^{'}} |f(t)|dt \\
&= \int_{\bigcup_{i=1}^{n}(x_i, x_i^{'})} |f(t)|dt \\
&= \int_{\bigcup_{i=1}^{n}(x_i, x_i^{'})} |f| \\
&< \epsilon.
\end{aligned}$$

Hence $F(x)$ is absolutely continuous. □

(iii) the sum and difference of two absolutely continuous functions

is absolutely continuous.

Proof. Let f, g be absolutely continuous functions.

Let $\epsilon > 0$; there is a $\delta > 0$ such that $\forall\{(x_i, x_i^{'})\}_{i=1}^{n}$ of nonoverlapping intervals in $[a, b]$ with $\sum_{i=1}^{n} |x_i^{'} - x_i| < \delta$ we have

$$\begin{aligned}
\sum_{i=1}^{n} |(f+g)(x_i^{'}) - (f+g)(x_i)| &= \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i) + g(x_i^{'}) - g(x_i)| \\
&\leq \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i)| + \sum_{i=1}^{n} |g(x_i^{'}) - g(x_i)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} |(f-g)(x_i^{'}) - (f-g)(x_i)| &= \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i) + g(x_i) - g(x_i^{'})| \\
&\leq \sum_{i=1}^{n} |f(x_i^{'}) - f(x_i)| + \sum_{i=1}^{n} |g(x_i^{'}) - g(x_i)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence the sum and difference of two absolutely continuous functions

is absolutely continuous. □

Lemma 11. *If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.*

Proof. Assume f is absolutely continuous on $[a, b]$.

Let $\epsilon = 1$; there is a $\delta > 0$ correspond to ϵ in the definition of absolutely continuity.

Claim $T < \infty$.

Let $a = x_0 < x_1 < ... < x_n = b$ be a subdivision of $[a, b]$.

Let $K = \lfloor 1 + (b-a)/\delta \rfloor$.

Then $a = x_0 < x_1 < ... < x_n = b$ can be split into K sets of

nonoverlapping intervals each of total length less than δ.

$$\begin{aligned}(i.e., a = x_{n_0} < x_{n_0+1} < \dots < x_{n_1-1} < x_{n_1} < x_{n_1+1} < \dots \\ < x_{n_2-1} < x_{n_2} < x_{n_2+1} < \dots \\ < x_{n_K-1} < x_{n_K} = b \\ for\ some\ n_j \in \mathbb{N},\ x_{n_j} \in [a,b] \\ where\ \forall\ j \in \{1,2,\dots,K\},\ \sum_{i=n_{j-1}}^{n_j-1} |x_{i+1} - x_i| < \delta.)\end{aligned}$$

Thus $t = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{j=1}^{K}(\sum_{i=1}^{n_j} |f(x_{j(i-1)}) - f(x_{ji})|) < K \cdot 1 = K$.

Hence $T = sup\ t \leq K$. □

Corollary 12. *If f is absolutely continuous, then f has a derivative almost everywhere.*

Proof. Assume f is absolutely continuous.

Let $a, b \in \mathbb{R}$. Then f is absolutely continuous on $[a,b]$.

By Lemma 11, f is of bounded variation on $[a,b]$.

By Corollary 6, $f^{'}(x)$ exists for almost all $x \in [a,b]$.

Hence f has a derivative almost everywhere. □

Lemma 13. *If f is absolutely continuous on $[a,b]$ and $f^{'}(x) = 0$ a.e., then f is constant.*

Proof. Let $c \in [a,b]$.

Want to show that $f(a) = f(c)$.

Since $f^{'}(x) = 0$ a.e.,$\exists N \subseteq [a, b]$ such that $m(N) = 0$

and $f^{'}(x) = 0$ for $x \in [a, b] \setminus N$.

Set $E = (a, c) \setminus N$.

Then $m(E) = c - a$ and $f^{'}(x) = 0$ for $x \in E$.

Let $\epsilon > 0$ and $\eta > 0$; there is a $\delta > 0$ corresponding to ϵ in

the definition of the absolute continuity of f.

Let $x \in E$.

Since $E \subseteq (a, c)$; there is a $h > 0$ such that $[x, x + h] \subseteq [a, c]$.

Since $f^{'}(x) = 0$, $|f(x + h) - f(x)| < \eta h$.

Set $\mathcal{C} = \{[x, y] \subseteq [a, c] \mid |f(y) - f(x)| < \eta(y - x)\}$.

Claim $\mathcal{C}$ covers E in the sense of Vitali.

Let $\epsilon > 0$ and $x \in E$.

Since $x \in E \subseteq (a, c)$ and $f^{'}(x) = 0$; there is a $0 < h < \epsilon$ such that

$[x, x + h] \subseteq [a, c]$ and $|f(x + h) - f(x)| < \eta h$.

Thus $[x, x + h] \in \mathcal{C},\ x \in E$ and $l([x, x + h]) = h < \epsilon$.

Hence $\mathcal{C}$ covers E in the sense of Vitali.

By Lemma 1(Vitali); there is a $\{[x_i, y_i]\}_{i=1}^{n}$ of nonoverlapping intervals in $\mathcal{C}$ such that $m(E \setminus \bigcup_{i=1}^{n} [x_i, y_i]) < \delta$.

WLOG, set $x_i < x_{i+1}$, we have

$$y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 \leq \ldots \leq x_n < y_n \leq c = x_{n+1}$$

and

$$\sum_{i=0}^{n} |x_{i+1} - y_i| = m(E \setminus \bigcup_{i=1}^{n} [x_i, y_i]) < \delta. \qquad (*)$$

Thus

$$\begin{aligned}\sum_{i=0}^{n} |f(y_i) - f(x_i)| &\leq \sum_{i=1}^{n} \eta(y_i - x_i) \\ &= \eta \sum_{i=1}^{n} (y_i - x_i) \\ &\leq \eta(c - a)\end{aligned}$$

and by (*), $\sum_{i=0}^{n} |f(x_{i+1}) - f(y_i)| < \epsilon$.

Therefore

$$\begin{aligned}|f(c) - f(a)| &= |\sum_{i=0}^{n} [f(x_{i+1}) - f(y_i)] + \sum_{i=1}^{n} [f(y_i) - f(x_i)]| \\ &\leq \sum_{i=0}^{n} |f(x_{i+1}) - f(y_i)| + \sum_{i=0}^{n} |f(y_i) - f(x_i)| \\ &< \epsilon + \eta(c - a).\end{aligned}$$

Since ϵ and η are arbitrary positive number, $f(c) - f(a) = 0$.

Hence f is constant. $\square$