5. Friction

2142111 Statics, 2011/2

© Department of Mechanical Engineering, Chulalongkorn University
Objectives  Students must be able to

Course Objective
- Include friction into equilibrium analyses

Chapter Objectives
For general dry friction
- Describe the mechanism and characteristics of dry friction, coefficient of friction and angle of friction
- Specify and determine the limiting conditions for sliding, toppling, free rolling and driven wheels under dry friction by appropriate FBDs
- Specify the limiting conditions for rear/front wheel drives by appropriate FBDs
- Analyze bodies/structures with friction for unknown loads/reactions by appropriate FBDs
Objectives  Students must be able to

Course Objective
- Include friction into equilibrium analyses

Chapter Objectives
For friction in machines
- Prove, state limitation and apply formula for impending motion in wedges, single continuous threads/screws, flat belts, disks/clutches, pivot/collar/thrust bearings, journal bearings
- Analyze machines with friction
Contents

- Dry friction
  - Characteristics, theory, coefficient and angle of friction
- Applications
  - Wedges
  - Threads, screws
  - Belts
  - Disks and Clutches
  - Collar, pivot, journal and thrust bearings
Friction

- Force of resistance acting on a body which prevents or retards slipping of the body relative to a surface with which it is in contact.
  - The frictional force acts tangent to the contacting surface in a direction opposed to the relative motion or tendency for motion of one surface against another.

- Two types of friction
  - Fluid friction exists when the contacting surfaces are separated by a film of fluid.
  - Dry friction occurs between contacting surfaces without lubricating fluid.
Dry Friction Theory #1

- Friction forces are tangential forces generated between contacting surfaces.
- Friction forces arise in part from the interactions of the roughnesses or asperities of the contacting surfaces.
Dry Friction

**Equilibrium**

- Slipping and/or tipping effects
- Frictional force $F$ increases with force $P$.  

\[ \Delta F_n \]

\[ \Delta N_n \]

Resultant Normal and Frictional Forces
Static Friction Equilibrium

- Maximum value of frictional force $F_s$ on the object in equilibrium is called the limiting static frictional force.
$F_s = \mu_s N$

$F_s$ = maximum static frictional force

$\mu_s$ = coefficient of static friction

$N$ = normal force

$F_s$ is the maximum friction forces that can be exerted by dry, contacting surfaces that are stationary relative to each other.
Static Friction Angle of Static Friction

\[ N = R_s \cos \phi_s \]
\[ F_s = R_s \sin \phi_s = \mu_s N \]
\[ \mu_s = \frac{F_s}{N} = \frac{R \sin \phi_s}{R \cos \phi_s} = \tan \phi_s \]

\[ \phi_s = \tan^{-1} \mu_s \]
### Static Friction Typical Values

<table>
<thead>
<tr>
<th>Contact Materials</th>
<th>$\mu_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metal / ice</td>
<td>0.03 – 0.05</td>
</tr>
<tr>
<td>Wood / wood</td>
<td>0.30 – 0.70</td>
</tr>
<tr>
<td>Leather / wood</td>
<td>0.20 – 0.50</td>
</tr>
<tr>
<td>Leather / metal</td>
<td>0.30 – 0.60</td>
</tr>
<tr>
<td>Aluminum / Aluminum</td>
<td>1.10 – 1.70</td>
</tr>
</tbody>
</table>

How to obtain values of $\mu_s$?
Frictional Forces

- More force is required to start sliding than to keep it sliding can be explained by the necessity to break these bonds.
Kinetic Friction Coefficient & Angle

Dry Friction

**Kinetic Friction**

\[ F_k = \mu_k N \]

- \( F_k \) = kinetic frictional force
- \( \mu_k \) = coefficient of kinetic friction
- \( N \) = normal force

\[ \phi_k = \tan^{-1} \mu_k \]
At $\phi = \phi_s$, slip is impending.
At $\phi = \phi_k$, surfaces are sliding relative to each other.
Dry Friction Characteristics

- Frictional force acts tangentially to the contacting surfaces, opposing the relative or tendency for motion.

- $F_s$ is independent of the area of contact, provided that the normal pressure is not very low nor great enough for deformation of the surfaces.

- In equilibrium:
  
  Impending slipping: $\phi = \phi_s$
  
  Slipping: $\phi = \phi_k$
  
  Very low velocity: $\phi_k \approx \phi_s$
Dry Friction

Motion

- Sliding
  - Relative sliding (translation motion) between two surfaces

- Toppling
  - Fall over (rotation) about the edge
  - Topple, tipping, rolling, tumble, trip
**Example Friction 1 #1**

Place a sheet of one materials on a flat board. Bond a sheet of other material to block $A$. Place the block onto the board, and slowly increase the tilt $a$ of the board until the block slips. Show that the coefficient of static friction between the two materials is related to the angle $a$ by $\mu_s = \tan \alpha$. 
Dry Friction

Example Friction 1 #2

The block is about to slip

\[
\begin{align*}
\sum F_y = 0 & \quad N - W \cos \alpha = 0 \\
& \quad N = W \cos \alpha \quad \text{(1)} \\
\sum F_x = 0 & \quad -F_s + W \sin \alpha = 0 \\
& \quad \mu_s N = W \sin \alpha \quad \text{(2)} \\
(2)/(1) & \quad \mu_s = \tan \alpha \quad \text{Ans}
\end{align*}
\]
Dry Friction

Example Friction 2 #1

Will this crate slide or topple over?
Dry Friction

Example Friction 2 #2

- The crate is about to slide.

\[ F = F_s = \mu_s N \]
Example Friction 2 #3

- The crate is about to topple.

\[ F < F_s \ (\mu_s N) \]

- The crate is about to slide and topple.

\[ F = F_s = \mu_s N \]
The uniform crate has a mass of 20 kg. If a force $P = 80 \text{ N}$ is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction $= 0.3$. 

\[ P = 80 \text{ N} \]
Dry Friction

Example Hibbeler Ex 8-1 #2

\[
\sum F_x = 0 \rightarrow + \\
(80 \text{ N}) \cos 30^\circ - F = 0 \rightarrow F = 69.3 \text{ N}
\]

\[
\sum F_y = 0 \uparrow + \\
-(80 \text{ N}) \sin 30^\circ - 196.2 \text{ N} + N_C = 0 \rightarrow N_C = 236.2 \text{ N}
\]

\[
\sum M_O = 0 \cup + \\
(80 \text{ N}) \sin 30^\circ (0.4 \text{ m}) - (80 \text{ N}) \cos 30^\circ (0.2 \text{ m}) + N_C(x) = 0 \rightarrow \\
x = -9.0753 \times 10^{-3} \text{ m} = -9.08 \text{ mm}
\]

\[|x| << 0.4 \text{ m} \rightarrow \text{The crate will not topple.}
\]

\[F < \mu_s N_C (70.8 \text{ N}) \rightarrow \text{The crate, is close to but doesn't not slip.}\]
Dry Friction

Example Hibbeler Ex 8-3 #1

The rod with weight $W$ is about to slip on rough surfaces at $A$ and $B$. Find coefficient of static friction.
Impending slip

\[ F_A = \mu_s N_A \]
\[ F_B = \mu_s N_B \]

3 equilibrium equations,
3 unknowns \((N_A, N_B, \mu_s)\)

\[
\begin{align*}
\sum F_x &= 0 \quad \rightarrow + \quad \mu_s N_A + \mu_s N_B \cos 30^\circ - N_B \sin 30^\circ = 0 \quad (1) \\
\sum F_y &= 0 \quad \uparrow + \quad N_A - W + N_B \cos 30^\circ + \mu_s N_B \sin 30^\circ = 0 \quad (2) \\
\sum M_A &= 0 \quad \odot + \quad N_B l - W \cos 30^\circ (l/2) = 0 \quad (3)
\end{align*}
\]
Dry Friction

**Example** Hibbeler Ex 8-3 #3

From (3) \( N_B = 0.4330W \)

From (1) & (2) \( 0.2165\mu_s^2 - \mu_s + 0.2165 \)

\( \mu_s = 0.228 \)  \( \text{Ans} \)
Example Hibbeler Ex 8-4 #1

Find minimum coefficients of static friction that keep the system in equilibrium.
Dry Friction

Example Hibbeler Ex 8-4 #2

- Symmetry of lower pipes about vertical axis passing through O.
- Two types of contacts:
  - Pipe – pipe
  - Pipe – ground

\[ N_A = N_B, \quad F_A = F_B, \]
\[ F_A \leq \mu_{s,p-p} N_A \]
\[ F_C \leq \mu_{s,p-g} N_C \]
Suppose that $\alpha = 10^\circ$ and the coefficient of friction between the surface of the wedge and the log are $\mu_s = 0.22$ and $\mu_k = 0.20$. Neglect the weight of the wedge.

- If the wedge is driven into the log at a constant rate by vertical force $F$, what are the magnitude of the normal forces exerted on the log by the wedge?
- Will the wedge remain in place in the log when the force is removed?
Dry Friction

Example  Bedford Ex 9.5 #2

Symmetry about the center of the wedge

\[
\sum F_y = 0 \quad \uparrow +
\]

\[
2N \sin(\alpha/2) + 2\mu_kN\cos(\alpha/2) - F = 0
\]

\[
N = \frac{F}{2\sin(\alpha/2) + 2\mu_k(\alpha/2)}
\]

\[
= \frac{F}{2\sin(10^\circ/2) + 2(0.20)(10^\circ/2)}
\]

\[
N = 1.75F \quad \text{Ans}
\]
Symmetry about the center of the wedge

\[
\sum F_y = 0 \quad \uparrow +
\]

\[2N \sin(\alpha / 2) + 2\mu_s N \cos(\alpha / 2) = 0\]
\[\mu_s' = \tan(\alpha / 2) = 0.087\]

As \(\mu_s' < \mu_s\), wedge will remain in place. Ans
The coefficient of static friction between the two boxes and between the lower box and the inclined surface is $\mu_s$. What is the largest angle $\alpha$ for which the lower box will not slip.
Dry Friction

Example Bedford 9.20 #2

FBD of upper block
\[ \sum F_y = 0 \]
\[ N_2 - W \cos \alpha = 0 \]
\[ N_2 = W \cos \alpha \]

FBD of lower block
\[ \sum F_y = 0 \]
\[ N_2 + W \cos \alpha - N_1 = 0 \]
\[ N_1 = 2W \cos \alpha \]
\[ \sum F_x = 0 \]
\[ \mu_s N_2 + \mu_s N_1 - W \sin \alpha = 0 \]
\[ \mu_s 3W \cos \alpha = W \sin \alpha \]
\[ \mu_s = \tan \alpha / 3 \] or \[ \alpha = \tan^{-1}(3 \mu_s) \]  \( \text{Ans} \)
Example Bedford 9.30 #1

The cylinder has weight $W$. The coefficient of static friction between the cylinder and the floor and between the cylinder and the wall is $\mu_s$. What is the largest couple $M$ that can be applied to the stationary cylinder without causing it to rotate.
Example Bedford 9.30 #2

Impending rotation

\[
\left[ \sum F_y = 0 \uparrow + \right] \quad \mu_s N_w + N_f - W = 0
\]
\[
\left[ \sum F_x = 0 \rightarrow + \right] \quad N_w - \mu_s N_f = 0
\]
\[
\left[ \sum M_O = 0 \uparrow + \right] \quad M - \mu_s (N_w + N_s) r = 0
\]

\[
N_w = \frac{\mu_s W}{1 + \mu_s^2}, \quad N_f = \frac{W}{1 + \mu_s^2}
\]

\[
M = \mu_s W r \frac{(1 + \mu_s)}{1 + \mu_s^2} \quad \text{Ans}
\]
The disk of weight \( W \) and radius \( R \) is held in equilibrium on the circular surface by a couple \( M \). The coefficient of static friction between the disk and the surface is \( \mu_s \). Show that the largest value \( M \) can have without causing the disk to slip is

\[
M = \frac{\mu_s RW}{\sqrt{1 + \mu_s^2}}
\]
Impending slip \( F_s = \mu_s N \)

\[
\begin{align*}
\sum F_x &= 0 \Rightarrow F_s - W \sin \alpha = 0 \\
\sum F_y &= 0 \Rightarrow N - W \cos \alpha = 0 \\
\sum M_o &= 0 \Rightarrow -M + F_s R = 0
\end{align*}
\]

\( \mu_s = \tan \alpha, \; N = W \cos \alpha \)

\[
\cos \alpha \sqrt{1 + \tan^2 \alpha} = 1, \; \tan \alpha = \mu_s
\]

\[
M = \mu_s N R = \mu_s W R \cos \alpha = \mu_s W R / \sqrt{1 + \mu_s^2}
\]
Each of the uniform 1-m bars has a mass of 4 kg. The coefficient of static friction between the bar and the surface at $B$ is 0.2. If the system is in equilibrium, what is the magnitude of the friction force exerted on the bar at $B$. 

![Diagram of the system with bars and angles]

---

**Example** Bedford 9.166 #1
Example Bedford 9.166 #2

\[
\begin{align*}
\sum F_x &= 0 \\
O_x + A_x &= 0 \\
\sum F_y &= 0 \\
O_y + A_y - 4g &= 0 \\
\sum M_o &= 0 \\
A_y (1 \text{ m}) \cos 45^\circ - A_x (1 \text{ m}) \sin 45^\circ - (4g \text{ N})(0.5 \text{ m}) \cos 45^\circ &= 0
\end{align*}
\]
Dry Friction

Example Bedford 9.166 #3

\[
\begin{align*}
\sum F_x &= 0 \\
-A_x + F \cos 30^\circ - N \sin 30^\circ &= 0 \\
\sum F_y &= 0 \\
-A_y - (4g \; \text{N}) + F \sin 30^\circ + N \cos 30^\circ &= 0 \\
\sum M_B &= 0 \\
A_y (1 \; \text{m}) \cos 45^\circ + A_x (1 \; \text{m}) \sin 45^\circ + (4g \; \text{N})(0.5 \; \text{m}) \cos 45^\circ &= 0
\end{align*}
\]

\[A_y = 0, \; A_x = -2g \; \text{N}, \; O_y = 4g \; \text{N}, \; O_x = 2g \; \text{N}, \; N = 438 \; \text{N}, \; F = 2.63 \; \text{N} \quad \text{Ans}\]
Example Bedford 9.166 #4

Dry Friction
Example Bedford 9.163 #1

The coefficient of static friction between the tires of the 1000-kg tractor and the ground and between the 450-kg crate and the ground are 0.8 and 0.3, respectively. Starting from the rest, what torque must the tractor’s engine exert on the rear wheels to cause the crate to move? The front wheels can turn freely.
Dry Friction

Example Bedford 9.163 #2

The crate is about to move.

\[ F = \mu_{s,c} N \]

\[
\begin{bmatrix}
\sum F_y = 0 \\
-(450g \ N) + N = 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sum F_x = 0 \\
P - F = 0
\end{bmatrix}
\]

\[ P = 0.3(450g \ N) = 135g \ N \]
### Example Bedford 9.163 #3

**Dry Friction**

![Diagram of a car with forces and moments acting on it.](image)

\[
\sum F_x = 0 \quad \Rightarrow -P + 2F_2 = 0 \quad \Rightarrow F_2 = 67.5 \text{g N}
\]

\[
\sum M_O = 0 \quad \Rightarrow (0.8 \text{ m})F_2 - T = 0 \quad \Rightarrow T = (0.8 \text{ m})(135 \text{g N}) = 1.06 \text{ kN} \cdot \text{m} \quad \text{Ans}
\]
Example Bedford 9.165 #1

The mass of the vehicle is 900 kg, it has rear-wheel drive, and the coefficient of static friction between its tires and the surface is 0.65. The coefficient of static friction between the crate and the surface is 0.4. If the vehicle attempts to pull the crate up the incline, what is the largest value of the mass of the crate for which it will slip up the incline before the vehicle’s tires slip?
Impending motion:

\[ F_2 = \mu_s t N_2 \]

\[ \sum F_x = 0 \]

\[ -P \cos 20^\circ + 2F_2 = 0 \]

\[ \sum F_y = 0 \]

\[ 2N_2 + 2N_1 - (900g \, N) - P \sin 20^\circ = 0 \]

\[ \sum M_O = 0 \]

\[ (900g \, N)(1 \, m) + P \cos 20^\circ(0.8 \, m) + P \sin 20^\circ(3.7 \, m) - 2N_2(2.5 \, m) = 0 \]

\[ P = 563.57g \, N, \ 2N_2 = 814.75g \, N, \ 2F_2 = 529.59g \, N, \ 2N_1 = 278.00g \, N \]
Dry Friction

**Example** Bedford 9.165 #3

The crate is about to move. \( F = \mu_s c N \)

\[
\sum F_y = 0
\]

\(-W \cos 20^\circ + N = 0\)

\[
\sum F_x = 0
\]

\(P - F - W \sin 20^\circ = 0\)

\(W = 1.3930P\) \(N = 7701.1\) N

\(m = W / g = 785\) kg

Ans
Dry Friction

**Example** Bedford 9.165 #4

What is the maximum tension in the cord that the tractor may tow a crate over a ramp with 45° angle of incline before the tractor topples.
Example  Bedford 9.165 #5

Impending topple: $N_1 = 0$

$\sum M_Q = 0$

$P(2.0 \text{ m})/\sqrt{2} - (900 \text{ g N})(1.5 \text{ m}) = 0$

$\sum F_y = 0$

$2N_2 - (900 \text{ g N}) - P/\sqrt{2} = 0$

$\sum F_x = 0$

$-P/\sqrt{2} + 2F_2 = 0$

$P = 954.59 \text{ g N}, 2N_2 = 1575 \text{ g N}, 2F_2 = 675 \text{ g N}$

$F_2 \leq \mu_{s,t}N_2$, tires do not slip.

$P = 955 \text{ g N}$  Ans
Friction in Machines

- Friction helps some machines function.

- Machines with frictions
  - Wedges
  - Threads, screws
  - Belts
  - Disks and clutches
  - Bearings
Wedges

- Wedges exert a large lateral forces from the faces as a result of small angle.
The block is about to raise.
The slip of the load and wedge are impending.

\[ F_2 = \mu_s N_2, \quad F_3 = \mu_s N_3 \]

\[
\begin{align*}
\sum F_x &= 0 \quad \rightarrow + \\
- N_3 + N_2 \sin \theta + \mu_s N_2 \cos \theta &= 0
\end{align*}
\]

\[
\begin{align*}
\sum F_y &= 0 \quad \uparrow + \\
N_2 \cos \theta - \mu_s N_2 \sin \theta - \mu_s N_3 - W &= 0
\end{align*}
\]
Wedges Analysis #2

Machine Friction

FBD of the wedge

\[
\sum F_x = 0 \rightarrow +
\]

\[-N_2 \sin \theta - \mu_s N_2 \cos \theta - \mu_s N_1 + P = 0\]

\[
\sum F_y = 0 \uparrow +
\]

\[N_1 - N_2 \cos \theta + \mu_s N_2 \sin \theta = 0\]

\[P = \frac{\left(1 - \mu_s^2\right) \tan \theta + 2 \mu_s}{\left(1 - \mu_s^2\right) - 2 \mu_s \tan \theta} W\]
Threads/Screws

- Assumption: the shaft has a single continuous thread.
\[ \theta = \tan^{-1} \left( \frac{l}{2\pi r} \right) \]

- \( \theta \) = lead angle
- \( l \) = pitch of the thread (lead)
- \( r \) = mean radius of the thread
Machine Friction

Threads Motion

\[ h \]

\[ r \]
Threads Moving Upwards

\[
\begin{align*}
\sum F_x &= 0 \rightarrow + \\
S - R \sin(\theta + \phi) &= 0 \\
\sum F_y &= 0 \uparrow + \\
R \cos(\theta + \phi) - W &= 0
\end{align*}
\]

\[
S = W \tan(\theta + \phi) \text{ and } M = Sr
\]

\[
M = Wr \tan(\theta + \phi)
\]

On the verge of rotating: \(\phi = \phi_s = \tan^{-1} \mu_s\)

Uniform upwards motion: \(\phi = \phi_k = \tan^{-1} \mu_k\)
Machine Friction

**Threads Moving Downwards #1**

\[ \sum F_x = 0 \quad \rightarrow + \]
\[ S' - R \sin(\theta - \phi) = 0 \]
\[ \sum F_y = 0 \quad \uparrow + \]
\[ R \cos(\theta - \phi) - W = 0 \]

\[ S' = W \tan(\theta - \phi) \quad \text{and} \quad M' = S'r \]

\[ M' = Wr \tan(\theta - \phi) \]

Downward screw motion \((\theta > \phi)\)

On the verge of rotating: \( \phi = \phi_s = \tan^{-1} \mu_s \)

Uniform downwards motion: \( \phi = \phi_k = \tan^{-1} \mu_k \)
Threads Moving Downwards #2

$M$ is removed: $W$ is supported by friction alone, providing that $\phi \geq \theta$

Self-locking screw ($\theta = \phi$) (on the verge of rotating downward)

Self-locking condition
Machine Friction

**Threads Moving Downwards #3**

\[ \sum F_x = 0 \rightarrow + \]
\[ -S'' + R \sin(\phi - \theta) = 0 \]

\[ \sum F_y = 0 \uparrow + \]
\[ R \cos(\phi - \theta) - W = 0 \]

\[ S'' = W \tan(\phi - \theta) \text{ and } M'' = S''r \]

Very rough surface \( \theta < \phi \):
Reversed motion.

\[ M'' = Wr \tan(\phi - \theta) \]
Example Wedges/Threads 1 #1

The mass of block A is 60 kg. Neglect the weight of the 5° wedge. The coefficient of kinetic friction between the contacting surfaces of the block A, the wedge, the table, and the wall is 0.4. The pitch of the threaded shaft is 5 mm, the mean radius is 15 mm, the coefficient of kinetic friction between the thread and the mating groove is 0.2. What couple must be exerted on the threaded shaft to raise the block A at a constant rate?
Machine Friction

Example Wedges/Threads 1 #2

Draw FBD of the mass $A$ and the wedge.

Then, follow the same procedure for wedge analysis:

$$F = \left(\frac{1 - \mu_k^2}{1 - \mu_{k1}^2}\right)\tan \alpha + 2\mu_{k1} W$$

Substitute for all known values:

$$W_L = 60g \text{ N}$$
$$W_w = 0 \text{ N}$$
$$\alpha = 5^\circ$$
$$\mu_{k1} = 0.4 \text{ for all non-thread contacts}$$

$$F = 667.50 \text{ N}$$
Machine Friction

Example Wedges/Threads 1 #3

\[ l = 5 \text{ mm} \]
\[ r = 15 \text{ mm} \]
\[ \mu_{k2} = 0.2 \text{ for thread/groove} \]
\[ \phi_k = \arctan(\mu_{k2}) = 11.310^\circ \]
\[ \theta = \arctan(l/2\pi r) = 3.0368^\circ \]

Draw FBD of the thread

Find \( M \) that will push the thread upwards:

\[ M = Fr \tan(\theta + \phi_k) \]
\[ M = (667.50 \text{ N})(0.015 \text{ m})\tan(3.0368^\circ + 11.310^\circ) \]
\[ M = 2.5609 \text{ N} \cdot \text{m} = 2.56 \text{ N} \cdot \text{m} \quad \text{Ans} \]
Flat Belts

- Involve slippage of flexible cables, belts and ropes over sheaves and drums.

\[ T_2 \quad T_1 \]

**motion**
Consider line element

\[ \sum F_t = 0 \]

\[ \mu dN + T \cos \frac{d\theta}{2} - (T + dT) \cos \frac{d\theta}{2} = 0 \]

\[ \mu dN = dT \cos \frac{d\theta}{2} \]

\[ \sum F_n = 0 \]

\[ dN - T \sin \frac{d\theta}{2} - (T + dT) \sin \frac{d\theta}{2} = 0 \]

\[ dN = 2T \sin \frac{d\theta}{2} + dT \sin \frac{d\theta}{2} \]
Machine Friction

Flat Belts Analysis #2

\[ \mu = \mu_s \text{ (motion impending)} \]

\[ \mu = \mu_k \text{ (ongoing movement)} \]

\[ d\theta \rightarrow 0, \cos \frac{d\theta}{2} \rightarrow 1: \]

\[ \mu dN = dT \cos \frac{d\theta}{2} \approx dT \]

\[ d\theta \rightarrow 0, \sin \frac{d\theta}{2} \rightarrow \frac{d\theta}{2}: \]

\[ dN = 2T \sin \frac{d\theta}{2} \approx T d\theta \]

\[ \frac{dT}{T} = \mu d\theta \quad \rightarrow \quad \int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^\beta d\theta \quad \rightarrow \quad \ln \frac{T_2}{T_1} = \mu \beta \]
Example Flat Belts 1 #1

Friction resists the lowering of \( m \):
\[
T_2 = mg
\]
\[
T_1 = P = mg / 6
\]
\[
\beta = 1.25(2\pi)
\]

A force \( P = mg/6 \) is required to lower the cylinder at a constant slow speed with the cord making 1.25 turns around the fixed shaft. Calculate the coefficient of friction \( \mu_s \) between the cord and the shaft.

\[
\left[ T_2 = T_1 e^{\mu\beta} \right]
\]

\[
mg = \frac{mg}{6} e^{2.5\pi \mu_s}
\]

\[
\mu_s = 0.228 \quad \text{Ans}
\]
Example Flat Belts 2 #1

Calculate the force \( P \) on the handle of the differential band brake that will prevent the flywheel from turning on its shaft to which the torque \( M = 150 \text{ N} \cdot \text{m} \) is applied. The coefficient of friction between the band and the flywheel is \( \mu = 0.40 \).
Friction resists the attempt of $M$ to rotate the flywheel in the clockwise direction:

\[
T_2 = T_1 e^{\mu \beta} \quad T_2 = T_1 e^{\mu (7\pi/6)} \quad (1)
\]

\[
\sum M_{\text{center}} = 0 \quad -(150 \text{ N} \cdot \text{m}) + (0.15 \text{ m})(T_2 - T_1) = 0
\]

\[
T_2 - T_1 = 1000 \text{ N} \quad (2)
\]

Solve (1) and (2) \( T_1 = 300 \text{ N}, \ T_2 = 1300 \text{ N} \)
Machine Friction

Example Flat Belts 2 #3

\[ \sum M_0 = 0 \ \theta^+ \]

\[-(0.15 \text{ m}) T_2 + (0.075 \text{ m})(T_1 \sin 30^\circ) + (0.45 \text{ m}) P = 0 \]

\[ P = 408 \text{ N} \quad \text{Ans} \]
Disks & Clutches

- Use to connect and disconnect two coaxial rotating shafts.
- The shaft can support a couple due to friction forces between the disks.
- Determine the couple by using coefficient of static friction in the flat-ended thrust equation.

Draw forces on the plane only.
Machine Friction

Disks & Clutches Analysis #1

\[ dA = 2\pi r \, dr \]

\[ \sum F_x = 0 \quad \Rightarrow \quad P - \int w \, dA = 0 \]

\[ P = \int_0^R p \, 2\pi r \, dr = p\pi R^2 \]

\[ \sum M_x = 0 \quad \Rightarrow \quad M - \mu \int w r \, dA = 0 \]

\[ M = \mu \int_0^R pr \, 2\pi r \, dr = \frac{2}{3} \mu p \pi R^3 \]

\[ M = \frac{2}{3} \mu PR \]
Disks & Clutches Worn Analysis #1

- Distribution of forces on worn plates are not uniformed.
- For example, linear distribution

\[ dA = 2\pi r \, dr \]

At \( r = R, w = 0 \)

\[ r = 0, \quad w = p \]

\[ w = p(R - r)/R \]
Disks & Clutches Worn Analysis #2

Consider the left plate

\[ \sum F_x = 0 \]

\[ P - \int w \, dA = 0 \]

\[ P = \int_0^R p (R - r) 2\pi r \, dr / R = \frac{\pi p}{3} R^2 \]

\[ \sum M_x = 0 \]

\[ M - \mu \int w r \, dA = 0 \]

\[ M = \mu \int_0^R pr (R - r) 2\pi r \, dr / R \]

\[ M = \frac{1}{2} \mu PR \]
Machine Friction

**Bearings** 

### Pivot Bearings

![Diagram of a pivot bearing with forces and friction](image)

**Equation:** 

\[ M = \frac{2}{3} \mu PR \]

- \( \mu = \mu_s \) (motion impending)
- \( \mu = \mu_k \) (ongoing movement)
Machine Friction

Bearings Collar Bearings

\[ M = \frac{2}{3} \mu P \frac{R_i^3 - R_o^3}{R_i^2 - R_o^2} \]

\( \mu = \mu_s \) (motion impending)

\( \mu = \mu_k \) (ongoing movement)
Machine Friction

Bearings  Thrust Bearings #1

\[
dA = 2\pi R \, ds = 2\pi R \left( \frac{dR}{\cos \theta} \right)
\]

\[
A = \int dA = \int_{R_i}^{R_o} \frac{2\pi R}{\cos \theta} \, dR = \frac{\pi \left( R_o^2 - R_i^2 \right)}{\cos \theta}
\]
Machine Friction

Bearings  Thrust Bearings #2

\[ \sum F_x = 0 \]
\[ P - pA \cos \theta = 0 \]
\[ p = \frac{P}{A \cos \theta} = \frac{P}{\pi \left( R_o^2 - R_i^2 \right)} \]

\[ \sum M_x = 0 \]
\[ M - \int R(\mu p dA) = 0 \]
\[ M = \mu \int_{R_i}^{R_o} R \frac{P}{\pi \left( R_o^2 - R_i^2 \right) \cos \theta} \cdot 2\pi R \ dR \]
\[ M = \frac{2\mu P}{3 \cos \theta} \left( \frac{R_o^3 - R_i^3}{R_o^2 - R_i^2} \right) \]
Machine Friction

Bearings  Journal Bearings #1
Machine Friction

Bearings Journal Bearings #2

\[ \sum M_z = 0 \quad \Leftrightarrow \]

\( M - P(R \sin \phi) = 0 \)

\( M = PR \sin \phi \)

\( \approx PR \tan \phi \)

\( M = PR \sin \phi \approx PR \mu \)
The 100-mm-diameter pulley fits loosely on a 10-mm-diameter shaft for which the coefficient of static friction is $\mu_s = 0.4$. Determine the minimum tension $T$ in the belt needed to (a) raise the 100-kg block and (b) lower the block. Assume that no slipping occurs between the belt and pulley and neglect the weight of the pulley.
Machine Friction

Example Hibbeler 8-11 #2

Raise the block

\[ r_f = r \sin \phi_s = (5 \text{ mm}) \sin(\tan^{-1} 0.4) = 1.8570 \text{ mm} \]

\[ \sum M_{P_2} = 0 \bigcirc + \]

\[ (981 \text{ N})(50 \text{ mm} + 1.8570 \text{ mm} - T(50 \text{ mm} - 1.86 \text{ mm})) = 0 \]

\[ T = 1056.8 \text{ N} \]

\[ T = 1.06 \text{ kN} \quad \text{Ans} \]
Machine Friction

Hibbeler Example 8-11 #3

Lower the block

\[ r_f = r \sin \phi_s = (5 \text{ mm}) \sin(\tan^{-1} 0.4) = 1.8570 \text{ mm} \]

\[ \sum M_{P_2} = 0 \quad \Rightarrow \]

\[ (981 \text{ N})(50 \text{ mm} - 1.86 \text{ mm}) - T(50 \text{ mm} + 1.86 \text{ mm}) = 0 \]

\[ T = 910.63 \text{ N} \]

\[ T = 911 \text{ kN} \quad \text{Ans} \]
Example Boresi 10.9 #1
A weight $W$ is lifted at a constant rate by applying a couple $Rd$ to the compound jackscrew.

- Derive a formula for $R$ in terms of the angle $\theta$, the weight $W$, the length $d$ of the lever rod, the pitch $p$ of the screw threads, the mean radius $r$ of the screw threads, and the coefficient of kinetic friction $\mu_k$ of the screw. The friction of the vertical wall guides is negligible.

- For $\theta = 15^\circ$, $W = 1000$ lb, $d = 20$ in, $r = 0.75$ in, $p = 0.333$ in, and $\mu_k = 0.15$, calculate $R$. 
Machine Friction

Example Boresi 10.9 #3

\[
\sum F_y = 0 \quad 2T \cos \theta - W = 0
\]
\[
W = 2T \cos \theta
\]  \quad (1)
Machine Friction

Example Boresi 10.9 #4

\[
\begin{align*}
\sum F_y &= 0 \quad \Rightarrow \quad Q \cos \theta - T \cos \theta = 0 \\
Q &= T \\
\sum F_x &= 0 \quad \Rightarrow \quad 2T \sin \theta - P = 0 \\
P &= 2T \sin \theta \\
(1) + (2) \quad &\Rightarrow \quad P = W \tan \theta
\end{align*}
\]

By symmetry, FBD of the left nut is mirror image of the right nut FBD.
Machine Friction

Example Boresi 10.9 #5

\[ M = Wr \tan(\phi_k + \alpha) \]

\[ Rd = 2Pr \tan(\phi_k + \alpha) \]

\[ R = 2(W \tan \theta)r \tan(\tan^{-1} \mu_k + \tan^{-1} \frac{p}{2\pi r}) / d \]

Substitute numbers in the equation

\[ R = 4.48 \text{ lb} \quad \text{Ans} \]
The axle of the pulley fits loosely in a 50-mm-diameter pinhole. If $\mu_k = 0.30$ between the pinhole and the pulley axle and 0.20 between the pulley and the cord, determine the minimum tension $T$ required to turn the pulley counterclockwise at a constant velocity if the block weighs 60 N. Neglect the weight of the pulley.
The shaft is about to rotate.

\[ \phi = \tan^{-1} \mu_{k,b} = \tan^{-1} 0.3 = 16.699^\circ \]

\[ r_f = r_2 \sin \phi = 7.1837 \text{ mm} \]

\[ \theta = \tan^{-1}(r_1 / r_1) = 45^\circ \]

\[ \beta = \sin^{-1}(r_f / \sqrt{2r_1}) = 2.9117^\circ \]

\[
\begin{align*}
\sum F_y &= 0 \\
-60 + R \sin(\theta + \beta) &= 0
\end{align*}
\]

\[
\begin{align*}
\sum F_x &= 0 \\
T - R \cos(\theta + \beta) &= 0
\end{align*}
\]

\[ R = 80.850 \text{ N}, \quad T = 54.192 \text{ N} \]
If the cord does not slip over the pulley, the minimum $\mu_{k,f1}$ is:

$$T_2 = T_1 e^{\mu_{k,f}\beta}$$

$$60 \text{ N} = (54.192 \text{ N}) e^{\mu_{k,f1}(\pi/2)}$$

$$\mu_{k,f1} = 0.064815$$

$\mu_{k,b1} < 0.2$, the assumption the the cord does not slip is valid.

$$T = 54.2 \text{ N}$$

\textbf{Ans}
The belt is about to slide.

\[
T_2 = T_1 e^{\mu_k \beta}
\]

\[
60 \text{ N} = Te^{0.2(\pi/2)}
\]

\[
T = 43.824 \text{ N}
\]
If the shaft does not rotate,

\[
\begin{align*}
\sum F_y &= 0 \\
-(60 \text{ N}) + R \sin(\theta + \beta) &= 0 \\
\sum F_x &= 0 \\
T - R \cos(\theta + \beta) &= 0
\end{align*}
\]

\( T = 43.824 \text{ N}, \ \theta = 45^\circ, \ \beta = 8.8556^\circ \)

\( \beta = \sin^{-1}(r_f / \sqrt{2}r_1) \rightarrow r_f = 21.771 \text{ mm} \)

\( r_f = r_2 \sin \phi \rightarrow \phi = 60.557^\circ \)

\( \phi = \tan^{-1} \mu_{k,b1} \rightarrow \mu_{k,b1} = 1.77 \)

\( \mu_{k,b1} > 0.3, \) the assumption that the shaft does not rotate is invalid.
Example Boresi 10-86 #1

A torque $T$ is applied to pulley $A$, which drives pulley $B$. The two pulleys have equal radii. The tension in the slack side of the belt is 2 kN. The coefficient of friction between the belt and the pulleys is 0.30.

- What is the maximum possible torque that can be applied to pulley $A$?
- What torque is transmitted to pulley $B$?
Machine Friction

**Example Boresi 10-86 #2**

Given $T_2 = 2 \text{ kN}$

The belt is about to slip:

$$T_1 = T_2 e^{\mu \beta}$$

$$T_1 = (2 \text{ kN}) e^{0.3 \pi} = 5.1327 \text{ kN}$$

$$T_1 = 5.13 \text{ kN} \quad \text{Ans}$$

$$\sum M_O = 0$$

$$-T + (T_1 - T_2)r = 0$$

$$T = (T_1 - T_2)(0.1 \text{ m}) = 0.31327 \text{ kN} \cdot \text{m}$$

As tensions in the belts remain unchanged between the two pulley,

$$T_B = T = 313 \text{ N} \cdot \text{m} \quad \text{Ans}$$
Example Boresi 10-98 #1

In some applications of collar bearings, multiple collars are used. Two collars are used as shown to support the 10,000 lb load ($\mu_s = 0.40$ and $\mu_k = 0.20$)

- Determine the torque $T$ required to initiate rotation of shaft $S$. Assume that the load is divided equally between the collars and that the pressure on each collar is uniform.
- Why are multiple collars used?
Example Boresi 10-98 #2

The load is divided equally between the collars, thus, the load on each collar is 5 kip.

For each collar that is about to rotate:

\[
T = \frac{2}{3} \mu_s P \left( \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2} \right)
\]

\[
T = \frac{2}{3} \times 0.4(5000 \text{ lb}) \left( \frac{(1 \text{ ft})^3 - (0.5 \text{ ft})^3}{(1 \text{ ft})^3 - (0.5 \text{ ft})^3} \right) = 1555.6 \text{ lb} \cdot \text{ft}
\]

Total required torque \( T = 2M = 3.11 \text{ kip} \cdot \text{ft} \quad \text{Ans} \)

Less load on each collar, less wear and tear. \( \text{Ans} \)
Concepts #1

- Friction is the *tangent resistance force* which prevents or retards slipping of the body relative to a surface with which it is in contact. Maximum friction occurs at the *verge of impending motion* which can be *slipping or toppling or both together*.

- Some machine functions by friction. To use the formula, the physical meanings underlying the friction in machines must be understood.