6. Virtual Work

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Objectives Students must be able to #1

Course Objective
- Use virtual work and energy methods in analyses of frictionless bodies/structures in equilibrium

Chapter Objectives
For virtual work
- Describe and determine virtual works and virtual motion (displacement/rotation) by forces and couples
- Determine active forces that maintain systems in equilibrium and draw the Active Force Diagram (AFD)
- Determine possible virtual motions and draw the Virtual Motion Diagram (VMD)
- Determine the degree of freedom (DoF) and virtual motion in terms of independent coordinates
Objectives  Students must be able to #2

For virtual work
- Use the principle of virtual work with AFD and VMD to analyze for unknown active loads or equilibrium position for systems in equilibrium

For potential energy
- Relate the work done with potential energy
- Describe and determine the potential energy, potential function and independent coordinates in a system
- Describe the conditions for stable/neutral/unstable stability of a system in equilibrium by the potential function
- Determine from potential function the stability of 1 DoF systems in equilibrium
Contents

- Virtual Work
  - Definition of Work
  - Principle of Virtual Work
  - Types of Forces
  - Degree of Freedom

- Potential Energy
  - Applications
Virtual Work

**Work By a Force**

\[ dU = \vec{F} \cdot d\vec{r} = F \, dr \, \cos \theta \]

- \( dU \) = work
- \( \vec{F} \) = force that done the work
- \( d\vec{r} \) = displacement
Virtual Work

**Work By a Couple**

\[
dU = F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = (Fr)d\theta
\]

\[dU = M \, d\theta\]

- \(M\) = magnitude of couple that do the work
- \(d\theta\) = small angle of rotation
Virtual Work Utilization

- Utilize the principle of virtual work
  - To determine active forces that maintain the system in equilibrium,
  - To determine the equilibrium positions,
  - To relate the work done by conservative forces with potential energy.
Virtual Work

Virtual Movements

- Imaginary or virtual movements is assumed and does not actually exist.
  - Virtual displacement $\delta r$
  - Virtual rotation $\delta \theta$
- Virtual movements are infinitesimally small and does not violate physical constraints.

Principle of virtual work is an alternative form of Newton’s laws that can analyze the system in equilibrium under work and energy concepts.
Virtual Work Constraints of Movements

\[ \delta U = \vec{F} \cdot \delta \vec{r} \]

\[ \delta U = M \cdot \delta \theta \]
Virtual Work Principle of Virtual Work

- Consider an object in equilibrium
- The virtual work done by all forces to move the object with a virtual displacement

\[ \delta U = \sum (\vec{F} \cdot \delta \vec{r}) \]
\[ = (\vec{F}_1 + \vec{F}_2 + \vec{F}_3) \cdot \delta \vec{r} \]

In equilibrium, \( \delta U = 0 \)
Types of Forces: Active and Reaction

- Active forces generate virtual work.
  - External forces – applied forces, gravitational forces
  - Internal forces – spring forces, viscous forces

- Reaction and constrain forces do not create virtual work
Conservative Forces Descriptions #1

- Conservative forces generate virtual work that depends only on the initial and final locations, but not on is the path.
  - Conservative forces are active forces.
  - Example: weight, elastic spring
Virtual Work

Conservative Forces Descriptions #2

Virtual Displacement:

\[ V_g - W_y \]

Virtual Work:

\[ U = W_y \]

Datum

\[ y = ds \]

\[ dy = ds \cos \theta \]
Virtual Work

Conservative Forces Descriptions #3

\[ U = -\left( \frac{1}{2} ks_2^2 - \frac{1}{2} ks_1^2 \right) \]
Conservative System

- A conservative system is a system in which work done by a force is
  - Independent of path, i.e. work done by conservative force,
  - Equal to the difference between the final and initial values of the energy function,
  - Completely reversible, i.e. no loss.
Frictional Force

- Friction exerts on a body by surface due to relative motion
  - Work done depends on the path; the longer the path, the greater the work.
- Friction is not conservative.
- Work done is dissipated from the body in the form of heat.
Virtual Work

Virtual Work Advantages

- Work done by reaction forces are always zero.
- Only active forces cause virtual works.
- If the structure is in equilibrium, or \( \delta U = 0 \), sum of virtual works done by all active forces are zero.
- For complex structures, all unknown reaction and constraint forces are ignored.
- Use the principle of virtual work to determine
  - Active forces that maintain the system in equilibrium,
  - Equilibrium positions.
Virtual Work

**AFD Active Force Diagrams**

- An AFD shows only active forces in the system.
Virtual Work

DOF Degrees of Freedom

- DOFs are independent coordinates used to describe positions of the system.
- Virtual displacements can be derived in terms of DOFs.
Apply the principle of virtual work to derive a formula in terms of $a$, $b$, and $W$ for the magnitude $P$ of the force required for equilibrium of the bell crank $ABC$. The pin at $B$ is frictionless. Neglect the weight of the bell crank.

- Procedure
  - Active force diagram
  - Geometric relations of virtual movements
  - Virtual work equation

AFD of the rod 1 DOF system
Virtual Work

Example Boresi 12-35 #2

Virtual motion

\[ \delta r_C = b \delta \theta \]
\[ \delta r_A = a \delta \theta \]

Virtual Work

\[ \delta U = 0 \]
\[ P \delta r_A - W \delta r_C = 0 \]
\[ P a \delta \theta - W b \delta \theta = 0 \]
\[ P = b W / a \quad \text{Ans} \]

DIY: Check by FBD
**Example** Hibbeler 11-13 #1

The thin rod of weight $W$ rest against the smooth wall and floor. Determine the magnitude of force $P$ needed to hold it in equilibrium for a given angle $\theta$. 

**AFD of the rod**

1 DOF system

![Diagram of the rod against the wall and floor with a force $P$ and weights $W$.]
Virtual movements:

\[ \delta y_C + y_C = \frac{l}{2} \sin(\theta + \delta \theta) \]

\[ \delta y_C + \frac{l}{2} \sin \theta = \frac{l}{2} (\sin \theta \cos \delta \theta + \cos \theta \sin \delta \theta) \]

\[ \cos \delta \theta \rightarrow 1, \ \sin \delta \theta \rightarrow \delta \theta \]

\[ \delta y_C = \frac{l}{2} \delta \theta \cos \theta \]

\[ \delta x_A + x_A = l \cos(\theta + \delta \theta) \]

\[ \delta x_A = l (\cos \theta \cos \delta \theta - \sin \theta \sin \delta \theta - \cos \theta) \]

\[ \cos \delta \theta \rightarrow 1, \ \sin \delta \theta \rightarrow \delta \theta \]

\[ \delta x_A = -l \delta \theta \sin \theta \] (as \ \theta \ increase, \ x_A \ decrease)
Virtual Work

**Example** Hibbeler 11-13 #3

\[
\delta U = 0
\]

\[
P \delta x_A - W \delta y_C = 0
\]

\[
P l \delta \theta \sin \theta - W \frac{l}{2} \delta \theta \cos \theta = 0
\]

\[
P = \frac{1}{2} W \frac{\cos \theta}{\sin \theta}
\]

*Ans*

DIY: Check by FBD
Example Hibbeler 11-13 #4

Virtual movements: Alternative method

\[ y_c = \frac{1}{2} l \sin \theta \quad \rightarrow \quad \frac{dy_c}{d\theta} = \frac{1}{2} l \cos \theta \]

\[ \rightarrow \delta y_c = \frac{1}{2} l \cos \theta \delta \theta \]

\[ x_A = l \cos \theta \quad \rightarrow \quad \frac{dx_A}{d\theta} = -l \sin \theta \]

\[ \rightarrow \delta x_A = -l \sin \theta \delta \theta \]

Be careful of the sign!
Using the principle of virtual work, determine the relationship between the force $P$ and $Q$, in terms of $a$, $b$, $c$, and $d$, for the frictionless mechanism that is in equilibrium in the position shown. Neglect the weights of the bars.

AFD of the system

1 DOF system
Virtual Work

Example Boresi 12-38 #2

\[
\delta_1 = d \delta \theta, \quad \delta_2 = c \delta \theta, \quad \delta_3 = \delta_2 = c \delta \theta, \\
\frac{\delta_3}{a} = \frac{\delta_4}{a + b} \rightarrow \delta_4 = \frac{a + b}{a} c \delta \theta
\]
Virtual Work

Example Boresi 12-38 #3

\[ \delta U = 0 \]

\[-P \delta_1 + Q \delta_4 = 0\]

\[
P = \frac{\delta_4}{\delta_1} = \frac{a + b}{a} c \delta \theta \]

\[
\frac{P}{Q} = \frac{d}{d} (1 + \frac{b}{a}) \text{ Ans}
\]
A pulley-crank mechanism is used to raise the 400 lb weight. Using the principle of virtual work, determine the force $P$. Neglect the weight of the pulley.
Virtual Work

Example Boresi 12-36 #2

\[
\delta_1 = (2 \text{ ft})\delta\theta \\
\delta_2 = (1 \text{ ft})\delta\theta \\
\delta_3 = \delta_4 \\
\quad = \delta_2 / 2 = (0.5 \text{ ft})\delta\theta
\]

\[
[\delta U = 0] \\
P\delta_1 - (400 \text{ lb})\delta_4 = 0 \\
P = (400 \text{ lb}) \frac{(0.5 \text{ ft})\delta\theta}{(2 \text{ ft})\delta\theta} \\
P = 100 \text{ lb} \quad \text{Ans}
\]

AFD of the system
1 DOF system
Virtual Work

Example Frame & Machine 1 #1

For $P = 150$ N squeeze on the handles of the pliers, determine the force $F$ applied by each jaw.
Virtual Work

Example Frame & Machine 1 #2

1 DOF system:

\[
\frac{\delta_1}{180 \text{ mm}} = \frac{\delta_2}{30 \text{ mm}} \rightarrow \delta_2 = \frac{\delta_1}{6}
\]

\[
\frac{\delta_2}{60 \text{ mm}} = \frac{\delta_3}{20 \text{ mm}} \rightarrow \delta_3 = \frac{\delta_2}{3} = \frac{\delta_1}{18}
\]
Virtual Work

**Example Frame & Machine 1 #3**

Symmetry about the horizontal centerline

\[
[\delta U = 0] \quad 2P\delta_1 - 2F\delta_3 = 0
\]

\[
F = 18P = 2.25 \text{ kN} \quad \text{Ans}
\]
Virtual Work

**Example Frame & Machine 2 #1**

The mechanism is used to weigh mail. A package placed at A causes the weighted pointer to rotate through an angle $\alpha$. Neglect the weights of the members except for the counterweight at $B$, which has a mass of 4 kg. If $\alpha = 20^\circ$, what is the mass of the package at A?
Virtual Work

**Example** Frame & Machine 2 #2

1 DOF system

B rotates CCW by $\delta \theta$.

$\delta_2 = (100 \text{ mm})(\cos(20^\circ - \delta \theta) - \cos 20^\circ) = (100 \text{ mm})\sin 20^\circ \delta \theta$

$\delta_1 = (100 \text{ mm})(\sin(10^\circ + \delta \theta) - \sin 10^\circ) = (100 \text{ mm})\cos 10^\circ \delta \theta$

$[\delta U = 0]$

$4g\delta_2 - W\delta_1 = 0$

$W = 4g \frac{\sin 20^\circ}{\cos 10^\circ}$, $m_A = 1.39 \text{ kg}$

**Ans**
Virtual Work Systems with Friction

- Friction forces causes the virtual work on the systems.
- As the advantage of virtual work method is in the analysis of the entire system, appreciable friction in the system requires the dismembering and negates this advantage.
Virtual Work

**Mechanical Efficiency**

\[ e = \frac{\text{output work}}{\text{input work}} \]

\[ 1 = \frac{\text{output work} + \text{loss}}{\text{input work}} \]
Find the mechanical efficiency in moving the block of weight $W$ up the incline with coefficient of kinetic friction $\mu_k$. 
The block moves upwards.
By equilibrium of force
\[ F = \mu_k W \cos \theta \]
$\delta U = 0$

$T \delta s - F \delta s - W \delta s \sin \theta = 0$

$T = W (\mu_k \cos \theta + \sin \theta)$

$e = \frac{W \delta s \sin \theta}{T \delta s} = \frac{\sin \theta}{\mu_k \cos \theta + \sin \theta}$
Potential Energy

- Potential energy
  - Capacity to do work
  - Generated by conservative forces

- Example of potential energy
  - Gravitational $V_g = mgh$
  - Elastic spring $V_e = 0.5ks^2$
  - Torsional spring $V_e = 0.5K\theta^2$

- Potential function $V = V_g + V_e$
Position from datum is defined by independent coordinate $q$

The displacement of frictionless system is $dq$

- $V = V(q)$
- $dU = V(q) - V(q + dq)$

In equilibrium $dU = -dV = 0$

$$\frac{dV}{dq} = 0$$
**Potential Energy**

**Equilibrium \( n \) DOFs**

- Positions defined by \( q_1, q_2, q_3, \ldots, q_n \)
- The displacement of frictionless system is \( \delta q_1, \delta q_2, \delta q_3, \ldots, \delta q_n \)
  - \( V = V(q_1, q_2, q_3, \ldots, q_n) \)
  - \( dU = V(q_1, \ldots, q_n) - V(q_1 + dq_1, \ldots, q_n + dq_n) \)

- In equilibrium \( dU = -dV = 0 \)
  
  \[
  \delta V = \frac{\partial V}{\partial q_1} \delta q_1 + \frac{\partial V}{\partial q_2} \delta q_2 + \ldots + \frac{\partial V}{\partial q_n} \delta q_n
  \]

  \[
  \frac{\partial V}{\partial q_1} = 0, \quad \frac{\partial V}{\partial q_2} = 0, \quad \ldots, \quad \frac{\partial V}{\partial q_n} = 0,
  \]
Potential Energy

Stability

A

Stable equilibrium

B

Neutral equilibrium

C

Unstable equilibrium

Position 1

Position 2

Position 3
Stability 1 DOF #1

\[ \frac{dV}{dq} = 0, \quad \frac{d^2V}{dq^2} > 0 \]

Stable equilibrium
Stability 1 DOF #2

Potential Energy

Neutral Equilibrium

\[ \frac{dV}{dq} = 0, \quad \frac{d^2V}{dq^2} = 0 \]

\[ \frac{d^2V}{dq^2} = 0 \Rightarrow \frac{dV}{dq} = 0 \]

Neutral equilibrium
Potential Energy

Stability 1 DOF #3

Unstable equilibrium

Unstable Equilibrium

\[
\frac{dV}{dq} = 0, \quad \frac{d^2V}{dq^2} < 0
\]
Determine the equilibrium position $s$ for the 5-lb block and investigate the stability at this position. The spring is unstretched when $s = 2$ in. and the inclined plane is smooth.
Example Hibbeler 11-31 #2

Potential Energy

Unstretched position

Equilibrium position

2 in.

30°

s

30°
Potential Energy

Example Hibbeler 11-31 #3

Potential function

\[ V = V_e + V_g = \frac{1}{2} ks^2 + mgh \]

\[ = \frac{1}{2} \left(3 \text{ lb/in.}\right) (s - 2 \text{ in.})^2 \]

\[ - (5 \text{ lb})(s - 2 \text{ in.}) \sin 30^\circ \]

\[ = \frac{1}{2} \left(3 \text{ lb/in.}\right) (s - 2 \text{ in.})^2 \]

\[ - \frac{(5 \text{ lb})}{2} (s - 2 \text{ in.}) \]

At equilibrium, \([dV / ds = 0] \]

\[ (3 \text{ lb/in.})(s - 2 \text{ in.}) - \frac{(5 \text{ lb})}{2} = 0 \]

\[ s = 2.8333 \text{ in.} \]
Example Hibbeler 11-31 #4

Stability consideration: \( \frac{d^2V}{ds^2} = (3 \text{ lb/in.}) \rightarrow \text{stable} \)

Stable equilibrium position \( s = 2.83 \text{ in.} \)  \( \text{Ans} \)
The 2-lb semi-cylinder supports the block which has a specific weight of $\gamma = 80 \text{ lb/ft}^3$. Determine the height $h$ of the block which will produce neutral equilibrium in the position shown.


\[ V = V_{g1} + V_{g2} = \gamma V_1 h_{G1} + W_2 h_{G2} \]

\[ = \left( \frac{80}{12^3} \text{lb/in.}^3 \right) (8 \text{ in.})(10 \text{ in.})h \left( 4 \text{ in.} + \frac{h}{2} \cos \theta \right) + (2 \text{ lb})(4 \text{ in.} - \frac{16 \text{ in.}}{3\pi} \cos \theta) \]

\[ = (14.816 \text{ lb})h + (1.8549 \text{ lb/in.})h^2 \cos \theta + (8 - 3.3953 \cos \theta) \text{lb \cdot in.} \]
Potential Energy

**Example Hibbeler 11-45 #3**

\[
\frac{dV}{d\theta} = -(1.8549 \text{ lb/in.})h^2 \sin \theta + (3.3953 \text{ lb\cdot in.})\sin \theta
\]

\[
\frac{d^2V}{d\theta^2} = -(1.8549 \text{ lb/in.})h^2 \cos \theta + (3.3953 \text{ lb\cdot in.})\cos \theta
\]

At equilibrium
\[
\begin{bmatrix}
\frac{dV}{d\theta} = 0 \\
\frac{d^2V}{d\theta^2} = 0
\end{bmatrix}
\]
\[-(1.8549 \text{ lb/in.})h^2 \sin \theta + (3.3953 \text{ lb\cdot in.})\sin \theta = 0\]
\[\theta = 0^\circ \text{ or } h = 1.3529 \text{ in.}\]

At neutral stability
\[
\begin{bmatrix}
\frac{d^2V}{d\theta^2} = 0 \\
\frac{d^2V}{d\theta^2} = 0
\end{bmatrix}
\]
\[-(1.8549 \text{ lb/in.})h^2 \cos \theta + (3.3953 \text{ lb\cdot in.})\cos \theta = 0\]
\[\theta = 90^\circ \text{ or } h = 1.3529 = 1.35 \text{ in.} \quad \text{Ans}\]
Example Boresi 12-44 #1

The uniform slender rods of equal length are hinged together. They rest against a 45° inclined plane and are constrained to remain in a vertical plane.

- Neglect friction, determine the smallest nonzero angle $\theta$ for which equilibrium is possible.
- Determine whether or not the equilibrium state is stable for this configuration.
Example Boresi 12-44 #2

AFD of the system
1 DOF system

\[ h_1 = a \sin \alpha \]
\[ h_2 = 2a \sin \alpha + a \cos \alpha \]
\[ \alpha = 45^\circ + \theta \]
\[ V = Wh_1 + Wh_2 \]
\[ = 3Wa \sin \alpha + Wa \cos \alpha \]
\[ \frac{dV}{d\theta} = Wa(3 \cos \alpha - \sin \alpha) \]
\[ \frac{d^2V}{d\theta^2} = Wa(-3 \sin \alpha - \cos \alpha) \]
Example Boresi 12-44 #3

In equilibrium
\[ \frac{dV}{d\theta} = Wa(3\cos \alpha - \sin \alpha) = 0 \]
\[ \alpha = 71.565^\circ, \ \theta = 26.6^\circ \quad \text{Ans} \]

Stability at \( \alpha = 71.565^\circ \)
\[ \frac{d^2V}{d\theta^2} = Wa(-3\sin \alpha - \cos \alpha) \]
\[ = -3.1623Wa \]
System is unstable \quad \text{Ans}
Concepts

The virtual work done by all forces to move the object with a *virtual displacement*. If the structure is in equilibrium, *sum of virtual works done by all active forces are equal to zero*.

Type of the system equilibrium – *stable, neutral* and *unstable*, can be determined by the second derivation of potential function with the independent coordinate.