2.4 Bases and Dimensions

Let V be a vector space over F. A subset $\mathcal{B} \subset V$ is a **basis** for V if $\mathcal{B}$ is linearly independent and $\operatorname{Span} \mathcal{B} = V$.

Theorem 2.4.1. *Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V. Then*

$$\forall\, \vec{v} \in V, \exists!\, a_1, \ldots, a_n \in F, \vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

Proof. Let $\vec{v} \in V$. Since $\operatorname{Span} \mathcal{B} = V$,

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

for some $a_1, \ldots, a_n \in F$. For uniqueness, let $b_1, \ldots, b_n \in F$ be such that

$$\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n.$$

Then

$$\begin{aligned} a_1\vec{v}_1 + \cdots + a_n\vec{v}_n &= b_1\vec{v}_1 + \cdots + b_n\vec{v}_n \\ (a_1 - b_1)\vec{v}_1 + \cdots + (a_n - b_n)\vec{v}_n &= \vec{0}. \end{aligned}$$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, we have

$$a_1 - b_1 = \cdots = a_n - b_n = 0,$$

so $a_i = b_i$ for all $i \in \{1, \ldots, n\}$. $\square$

Lemma 2.4.2. *If S is a linearly independent subset of V, then $\forall \vec{u} \in V \setminus \operatorname{Span} S$, $S \cup \{\vec{u}\}$ is linearly independent.*

Proof. If $V = \operatorname{Span} S$, then $V \setminus S = \emptyset$ and the statement is vacuously true.
Assume that $\operatorname{Span} S \subsetneq V$ and let $\vec{u} \in V \setminus \operatorname{Span} S$.
We shall show that $S \cup \{\vec{u}\}$ is linearly independent.
Let $c_1, \ldots, c_n \in F$ and $\vec{u}_1, \ldots, \vec{u}_n \in S$ be such that

$$c\vec{u} + c_1\vec{u}_1 + \cdots + c_n\vec{u}_n = \vec{0}.$$

If $c \neq 0$, then

$$\vec{u} = -\frac{1}{c}(c_1\vec{u}_1 + \cdots + c_n\vec{u}_n)$$

is in $\operatorname{Span} S$, which is a contradiction. Thus, $c = 0$, so

$$c_1\vec{u}_1 + \cdots + c_n\vec{u}_n = \vec{0}.$$

But S is linearly independent, we get $c_1 = \cdots = c_n = 0$. $\square$

Theorem 2.4.3. *Let V be a vector space over F.*
(1) If $\mathcal{B}$ is a linearly independent subset of V which is maximal with respect to the property of being linearly independent (i.e., $\forall \mathcal{B} \subseteq S, S \neq \mathcal{B} \Rightarrow S$ is not linearly independent), then $\mathcal{B}$ is a basis of V.
(2) If $\mathcal{B}$ is a spanning set for V which is minimal with respect to the property of spanning (i.e., $\forall S \subseteq \mathcal{B}, S \neq \mathcal{B} \Rightarrow \operatorname{Span} S \subsetneqq V$), then $\mathcal{B}$ is a basis of V.

Proof. (1) Assume that $\mathcal{B}$ is a maximal linearly independent subset of V.
It remains to prove that $\operatorname{Span}\mathcal{B} = V$. Let $\vec{v} \in V \smallsetminus \operatorname{Span}\mathcal{B}$.
By Lemma 2.4.2, $\mathcal{B} \cup \{\vec{v}\}$ is linearly independent which contradicts the maximality of $\mathcal{B}$. Hence, $\operatorname{Span}\mathcal{B} = V$.
(2) Let $\mathcal{B}$ be a minimal spanning set for V.
We shall prove that $\mathcal{B}$ is linearly independent.
Let $c_1, \ldots, c_n \in F$ and $\vec{v}_1, \ldots, \vec{v}_n \in \mathcal{B}$ be such that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Assume that $c_i \neq 0$ for some i. Then

$$\vec{v}_i = -\frac{1}{c_i}(c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n) \in \operatorname{Span}(\mathcal{B} \smallsetminus \{v_i\}).$$

Thus, $V = \operatorname{Span}\mathcal{B} = \operatorname{Span}(\mathcal{B} \smallsetminus \{v_i\})$ which contradicts the minimality of $\mathcal{B}$.
Hence, $\mathcal{B}$ is linearly independent. □

Theorem 2.4.4. *Let V be a vector space over F. If V has a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, then any linearly independent set in V has $\leq m$ elements.*

Proof. Assume that $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V$.
Let S be a linearly independent subset of V. Assume that $|S| > m$.
Let $T = \{\vec{u}_1, \ldots, \vec{u}_{m+1} \subseteq S \subset V$.
Since $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V$, we have

$$\begin{aligned}
\vec{u}_1 &= c_{11}\vec{v}_1 + c_{21}\vec{v}_2 + \cdots + c_{m1}\vec{v}_m \\
\vec{u}_2 &= c_{12}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{m2}\vec{v}_m \\
&\vdots \\
\vec{u}_{m+1} &= c_{1,m+1}\vec{v}_1 + c_{2,m+1}\vec{v}_2 + \cdots + c_{m,m+1}\vec{v}_m
\end{aligned}$$

for some $c_{ij} \in F$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m+1\}$.
By Theorem 1.2.7, the matrix equation

$$\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1,m+1} \\ c_{21} & c_{22} & \ldots & c_{2,m+1} \\ \ldots & \ldots & & \ldots \\ c_{m1} & c_{m2} & \ldots & c_{m,m+1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \end{bmatrix} = \vec{0}_m$$

has a nontrivial solution $\vec{x} \neq \vec{0}_{m+1}$. Note that

$$\begin{aligned}
&x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_{m+1}\vec{u}_{m+1} \\
&= x_1(c_{11}\vec{v}_1 + c_{21}\vec{v}_2 + \cdots + c_{m1}\vec{v}_m) + x_2(c_{12}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{m2}\vec{v}_m) \\
&\quad + \cdots + x_{m+1}(c_{1,m+1}\vec{v}_1 + c_{2,m+1}\vec{v}_2 + \cdots + c_{m,m+1}\vec{v}_m) \\
&= (x_1c_{11} + x_2c_{12} + \cdots + x_{m+1}c_{1,m+1})\vec{v}_1 + (x_1c_{21} + x_2c_{22} + \cdots + x_{m+1}c_{2,m+1})\vec{v}_2 \\
&\quad + \cdots + (x_1c_{m1} + x_2c_{m2} + \cdots + x_{m+1}c_{m,m+1})\vec{v}_m \\
&= \vec{0}
\end{aligned}$$

which contradicts S is linearly independent. Hence, $|S| \leq m$. □

Corollary 2.4.5. *If the vector space V has a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, then*
(1) $\{\vec{v}_1, \ldots, \vec{v}_m\}$ has a subset which is a basis for V,
(2) any linearly independent set in V can be extended to a basis for V,
(3) V has a basis,
(4) any two bases for V have the same finite number of elements, necessarily $\leq m$.

In this case, we say that V is **finite-dimensional**, and the number of elements in a basis is called the **dimension** of V, written $\dim V$. If V has no finite spanning set, we say that V is **infinite-dimensional**.

Proof. (1) Let $\mathscr{S} = \{S \subseteq \{\vec{v}_1, \ldots, \vec{v}_m\} : \operatorname{Span} = V\}$.
Since $\{\vec{v}_1, \ldots, \vec{v}_m\} \in \mathscr{S}$, $\mathscr{S}$ is a nonempty finite set partially ordered by $\subseteq$.
Then $\mathscr{S}$ has a minimal element, say $\mathcal{B}$.
By Theorem 2.4.3 (2), $\mathcal{B}$ is a basis for V.
(2) Let A be a linearly independent subset of V.
If $\operatorname{Span} A = V$, then A is a basis for V. If not, let $\vec{v} \in V \setminus \operatorname{Span} A$ and $A_1 = A \cup \{\vec{v}_1\}$. By Lemma 2.4.2, A_1 is linearly independent.
If $\operatorname{Span} A_1 = V$, then A_1 is a basis for V. If not the case, we continue the same process, we get A_2 which is linearly independent.
Theorem 2.4.4 assures that the process must stop at A_k, where $|A_k| \leq m$ and A_k is a basis for V.
(3) Since $\emptyset$ is linearly independent, $\emptyset$ can be extended to a basis for V by (2).
Hence, V has a basis.
(4) Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for V. By Theorem 2.4.4, $|\mathcal{B}_1|$ and $|\mathcal{B}_2|$ are finite. Since $\operatorname{Span} \mathcal{B}_1 = V$ and $\mathcal{B}_2$ is linearly independent, we have $|\mathcal{B}_2| \leq |\mathcal{B}_1|$ by Theorem 2.4.4. Interchange the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$, we get $|\mathcal{B}_1| \leq |\mathcal{B}_2|$. Hence, $|\mathcal{B}_1| = |\mathcal{B}_2|$ □

Remark. The above proof is valid for a "finite" dimensional vector space. For a general (finite/infinite dimensional) vector space V, consider

$$\mathscr{S} = \{S \subseteq V : S \text{ is linearly independent}\}.$$

Then $\emptyset \in \mathscr{S}$. Partially order $\mathscr{S}$ by $\subseteq$. We can show that every chain in $\mathscr{S}$ has an upper bound. By Zorn's lemma, $\mathscr{S}$ has a maximal element, say $\mathcal{B}$. By Theorem 2.4.3 (1), $\mathcal{B}$ is a basis of V. Hence, every vector space has a basis.

Example 2.4.1. Extend $\{(1,1,1)\}$ to a basis of $\mathbb{R}^3$.

Remark. If $\vec{e}_i$ is the ith column of the identity matrix I_m, then $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$ is a basis for F^m, called the **standard basis**, and so $\dim F^m = m$.

Corollary 2.4.6. *If V is a vector space with $\dim V = n$, then:*
(1) any spanning set of n elements is a basis of V
(2) any linearly independent set of n elements is a basis of V
(3) if W is an n-dimensional subspace of V, then $W = V$.

Proof. Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set of n vectors in V.
(1) Assume that $\operatorname{Span} S = V$. Then S has a subset which is a basis for V by Corollary 2.4.5 (1). But $\dim V = n$, so we have S is a basis for V.
(2) Assume that S is linearly independent. Then S can be extended to a basis for V by Corollary 2.4.5 (2). Again $\dim V = n$ implies S is a basis for V.
(3) Assume that W is a subspace of V and $\dim W = n$ with basis $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_n\}$. Then $\mathcal{B} \subseteq V$ and $\mathcal{B}$ is linearly independent.
But $|\mathcal{B}| = n = \dim V$, by (2), $\mathcal{B}$ is a basis for V. Hence, $W = \operatorname{Span} \mathcal{B} = V$. $\square$

Corollary 2.4.7. *If V is a finite-dimensional vector space and U is a proper subspace of V, then U is finite-dimensional and $\dim U < \dim V$.*

Proof. Assume that U is a proper subspace of V.
Let $\mathcal{B}$ be a basis of U. Then $\mathcal{B}$ is linearly independent.
By Corollary 2.4.5 (2), $\mathcal{B}$ can be extended to a basis $\mathcal{C}$ for V.
If $\mathcal{B} = \mathcal{C}$, then $U = V$ by Corollary 2.4.6.
Hence, $|\mathcal{B}| < |\mathcal{C}|$, so $\dim U < \dim V$. $\square$

Theorem 2.4.8. *If W_1 and W_2 are finite dimensional subspaces of a vector space V over a field F, then $W_1 + W_2$ is finite dimensional and*

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Proof. Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_r\}$ be a basis for $W_1 \cap W_2$.
Extend $\mathcal{B}$ to a basis $\mathcal{B}_1 = \{\vec{u}_1, \dots, \vec{u}_r, \vec{v}_1, \dots, \vec{v}_n\}$ for W_1
and to a basis $\mathcal{B}_2 = \{\vec{u}_1, \dots, \vec{u}_r, \vec{w}_1, \dots, \vec{w}_m\}$ for W_2.
Then $\dim W_1 = r + n$, $\dim W_2 = r + m$ and $\dim(W_1 \cap W_2) = r$.
We shall show that $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2 = \{\vec{u}_1, \dots, \vec{u}_r, \vec{v}_1, \dots, \vec{v}_n, \vec{w}_1, \dots, \vec{w}_m\}$ is a basis for $W_1 + W_2$. This implies that

$$\dim(W_1 + W_2) = r + n + m = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

as desired.

By Exercise Set II - 8, we have

$$\operatorname{Span}(\mathcal{B}_1 \cup \mathcal{B}_2) = \operatorname{Span} \mathcal{B}_1 + \operatorname{Span} \mathcal{B}_2 = W_1 + W_2.$$

Next, we prove linearly independence.

Let $c_1, \dots, c_r, d_1, \dots, d_n, e_1, \dots, e_m \in F$ be such that

$$c_1\vec{u}_1 + \cdots + c_r\vec{u}_r + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n + e_1\vec{w}_1 + \cdots + e_m\vec{w}_m = \vec{0}.$$

Then

$$c_1\vec{u}_1 + \cdots + c_r\vec{u}_r + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = -e_1\vec{w}_1 - \cdots - e_m\vec{w}_m,$$

so $-e_1\vec{w}_1 - \cdots - e_m\vec{w}_m \in W_1 \cap W_2$.
Since $W_1 \cap W_2 = \operatorname{Span}\{\vec{u}_1, \dots, \vec{u}_r\}$, there exist $c'_1, \dots, c'_r \in F$ such that

$$-e_1\vec{w}_1 - \cdots - e_m\vec{w}_m = c'_1\vec{u}_1 + \cdots + c'_r\vec{u}_r$$

Thus,

$$\begin{aligned} c_1\vec{u}_1 + \cdots + c_r\vec{u}_r + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n &= c'_1\vec{u}_1 + \cdots + c'_r\vec{u}_r \\ (c_1 - c'_1)\vec{u}_1 + \cdots + (c_r - c'_r)\vec{u}_r + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n &= \vec{0}. \end{aligned}$$

Since $\mathcal{B}_1$ is linearly independent, $c_i = c'_i$ for all $i \in 1, \dots, r$ and $d_j = 0$ for all $j \in \{1, \dots, n\}$. Therefore,

$$c_1\vec{u}_1 + \cdots + c_r\vec{u}_r + e_1\vec{w}_1 + \cdots + e_m\vec{w}_m = \vec{0}.$$

Since $\mathcal{B}_2$ is linearly independent, $c_i = 0$ for all $i \in 1, \dots, r$ and $e_k = 0$ for all $k \in \{1, \dots, m\}$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent. $\square$

Example 2.4.2. Consider two subspaces of $\mathbb{R}^5$

$$W_1 = \left\{ \begin{bmatrix} a \\ a-b \\ b \\ a+b \\ 0 \end{bmatrix} \in \mathbb{R}^5 : a, b \in \mathbb{R} \right\} \text{ and } W_2 = \left\{ \begin{bmatrix} c \\ d \\ 0 \\ e \\ d-e \end{bmatrix} \in \mathbb{R}^5 : c, d, e \in \mathbb{R} \right\}.$$

Find bases for W_1, W_2 and $W_1 \cap W_2$. Determine the dimension of $W_1 + W_2$.

Theorem 2.4.9. [Universal Mapping Property]
Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V.
If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$ (not necessarily distinct), then $\exists!$ linear transformation $T : V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$.

Proof. By Theorem 2.4.1, for each $\vec{v} \in V$ can be expressed uniquely in the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \text{ for some } c_1, c_2, \ldots, c_n \in F,$$

so if $T : V \to W$ is to be linear, we must define

$$T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n \text{ for all } \vec{v} \in V.$$

Then T is linear and $T(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$.
Finally, if $U : V \to W$ is another linear transformation such that $U(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$, then

$$\begin{aligned} U(\vec{v}) &= U(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) \\ &= c_1U(\vec{v}_1) + c_2U(\vec{v}_2) + \cdots + c_nU(\vec{v}_n) \quad \text{(because } U \text{ is linear)} \\ &= c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n \\ &= T(\vec{v}) \end{aligned}$$

for all $\vec{v} \in V$, and hence $U = T$. □

Example 2.4.3. (1) Find a linear transformation T that satisfies the following conditions

(a) $T : \mathbb{C} \to \mathbb{R}_2[x]$ with $T(1-i) = 2x^2$ and $T(1+i) = 1-x$,

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ with
$T(1) = (2,1), T(1-x) = (0,1)$ and $T(x+x^2) = (1,1)$.

(2) Let $T : \mathbb{R}_1[x] \to \mathbb{R}^3$ be a linear transformation with
$T(2-x) = (1,-1,1)$ and $T(1+x) = (0,1,-1)$.
Find $T(-1+2x)$.

Let V and W be vector spaces over a field and $T : V \to W$ a linear transformation. The **rank of** T, denoted by $\operatorname{rank} T$, is $\dim(\operatorname{im} T)$ and the **nullity of** T, denoted by $\operatorname{nullity} T$, is $\dim(\ker T)$.

Theorem 2.4.10. *Let V and W be vector spaces over a field F and $T : V \to W$ a linear transformation. If V is finite dimensional, then*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V.$$

Proof. Let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_k\}$ be a basis for $\ker T$.
Extend $\mathcal{B}$ to a basis $\mathcal{B}' = \{\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_m\}$ for V.
Then $T(\mathcal{B}') = \{\vec{0}_W, T(\vec{v}_1), \dots, T(\vec{v}_m)\}$.
By Exercise Set II - 19, we have

$$\operatorname{im} T = \operatorname{Span} T(\mathcal{B}') = \operatorname{Span}\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}.$$

We shall prove that the set $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is linear independent and hence it is a basis for $\operatorname{im} T$. Let $c_1, \dots, c_m \in F$ be such that

$$c_1 T(\vec{v}_1) + \cdots + c_m T(\vec{v}_m) = \vec{0}_m.$$

Then

$$T(c_1\vec{v}_1 + \cdots + c_m\vec{v}_m) = \vec{0}_m,$$

so $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in \ker T$. Thus,

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = d_1\vec{u}_1 + \cdots + d_k\vec{u}_k$$

for some $d_1, \dots, d_k \in F$, so

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + (-d_1)\vec{u}_1 + \cdots + (-d_k)\vec{u}_k = \vec{0}_V.$$

Since $\mathcal{B}'$ is linearly independent, we get

$$c_1 = \cdots = c_m = d_1 = \cdots = d_k = 0.$$

Hence, $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is linear independent. Therefore,

$$\operatorname{rank} T + \operatorname{nullity} T = \dim(\operatorname{im} T) + \dim(\ker T) = m + k = \dim V$$

as desired. $\square$

Theorem 2.4.11. *Let V and W be finite dimensional and $T : V \to W$ a linear transformation and $\dim V = \dim W$. Then T is 1-1 $\Leftrightarrow$ T is onto.*

Proof. Let T be a linear transformation.

$$\begin{aligned} T \text{ is 1-1} &\Leftrightarrow \ker T = \{\vec{0}_V\} \\ &\Leftrightarrow \operatorname{nullity} T = 0 \\ &\Leftrightarrow \operatorname{rank} T = \dim V \\ &\Leftrightarrow \operatorname{rank} T = \dim W \\ &\Leftrightarrow \dim(\operatorname{im} T) = \dim W \\ &\Leftrightarrow \operatorname{im} T = W \\ &\Leftrightarrow T \text{ is onto.} \end{aligned}$$

Hence, T is 1-1 $\Leftrightarrow$ T is onto. □

Corollary 2.4.12. *If V is finite dimensional, S and T are linear transformations from V to V, and $T \circ S$ is the identity map, then $T = S^{-1}$.*

Proof. We shall prove that S is $1-1$. Let $\vec{x} \in V$ be such that $S(\vec{x}) = \vec{0}_V$.
Since $T \circ S$ is the identity map, we have

$$\vec{x} = (T \circ S)(\vec{x}) = T(\vec{0}_V) = \vec{0}_V.$$

By Theorem 2.3.5, S is 1-1. Thus, S is onto by Theorem 2.4.11.
Hence, S is invertible and $T = S^{-1}$. □

From Theorem 2.4.1, we know that the representation of a given vector $\vec{v} \in V$ in terms of a given basis is unique. Let V be an n-dimensional vector space over a field F with an ordered basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $\vec{v} \in V$. Then

$$\forall \vec{v} \in V, \exists!(c_1, \ldots, c_n) \in F^n, \vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n.$$

The vector $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$ is called the **coordinate vector of $\vec{v}$ relative to the ordered basis $\mathcal{B}$**.

A *one-to-one* linear transformation from V *onto* W is called an **isomorphism**. If there exists an isomorphism from V onto W, then we say that V is **isomorphic** to W and we write $V \cong W$. Note that $\cong$ is an equivalence relation.

Theorem 2.4.13. *If V is an n-dimensional vector space over F, then $V \cong F^n$.*

Proof. Let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ be an ordered basis for V.
Define $T : V \to F^n$ by

$$\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n \mapsto \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{v}]_{\mathcal{B}}.$$

Let $\vec{v}, \vec{w} \in V$ and $a \in F$.
Then $\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$ and $\vec{w} = d_1\vec{u}_1 + \cdots + d_n\vec{u}_n$, so

$$[\vec{v} + \vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}$$

and

$$[a\vec{v}]_{\mathcal{B}} = \begin{bmatrix} ac_1 \\ \vdots \\ ac_n \end{bmatrix} = a \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = a[\vec{v}]_{\mathcal{B}}.$$

Hence, T is a linear transformation.

Next, we shall show that T is onto. Let $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$.

Choose $\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n \in V$. Then $T(\vec{v}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Since $\dim V = n = \dim F^n$, we have T is 1-1 by 2.4.11.

Therefore, T is an isomorphism and $V \cong F^n$. □

Corollary 2.4.14. *If V and W are finite dimensional, then*

$$\dim V = \dim W \Leftrightarrow V \cong W.$$

Let A be an $m \times n$ matrix. When A is row reduced to a matrix B, the column of B are often totally different from the columns of A. However, the equation $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have exactly the same set of solutions. That is, the column of A have exactly the same linear dependence relationships as the column of B. We conclude this result in

Theorem 2.4.15. *Elementary row operations on a matrix do not effect the linear dependence relations among the columns of the matrix. Moreover, the pivot columns of a matrix form a basis for* $\operatorname{Col} A$. *Hence,* $\dim(\operatorname{Col} A) = \operatorname{rank} A$.

For an $m \times n$ matrix A, $\dim(\operatorname{Nul} A)$, denoted by $\operatorname{nullity} A$, is called the **nullity of** A.

Recall that $T : \vec{x} \to A\vec{x}$ is a linear transformation from F^n to F^m.

Corollary 2.4.16. *Let A be an $m \times n$ matrix. Then*

$$\operatorname{nullity} A + \operatorname{rank} A = n.$$

Proof. It follows from Theorem 2.4.10. □

Example 2.4.4. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ -1 & 1 & -3 & 3 & 4 \\ 0 & 2 & -2 & 4 & 1 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}.$$

(i) Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$.

(ii) Find $\operatorname{rank} A$ and $\operatorname{nullity} A$.

Theorem 2.4.17. *Each $m \times n$ matrix A is row equivalent to a unique reduced echelon matrix U.*

Proof. The row reduction algorithm shows that there exists at least one such matrix U. Suppose that A is row equivalent to matrices U and V in reduced echelon form. The leftmost nonzero entry in a row of U is a "leading 1".

The pivot columns of U and V are precisely the nonzero columns that are *not* linearly dependent on the columns to their left. (This condition is satisfied automatically by a *first* column if it is nonzero.) Since U and V are row equivalent (both being row equivalent to A), their columns have the same linear dependence relations. Hence, the pivot columns of U and V appear in the same locations. If there are r such columns, then since U and V are in reduced echelon form, their pivot columns are the first r columns of the $m \times m$ identity matrix. Thus, *corresponding pivot columns of U and V are equal.*

Finally, consider any nonpivot column of U, say column j. This column is either zero or a linear combination of the pivot columns to its left (because those pivot columns are a basis for the space spanned by the columns to the left of column j). Either case can be expressed by writing $U\vec{x} = \vec{0}$ for some $\vec{x}$ whose jth entry is 1. Then $V\vec{x} = \vec{0}$, too, which says that column j of V is either zero or the *same* linear combination of the pivot columns of V to *its* left. Since corresponding pivot columns of U and V are equal, columns j of U and V are also equal. This holds for all nonpivot columns, so $V = U$, which proves that U is unique. $\square$