2.4 Bases and Dimensions

Let $V$ be a vector space over $F$. A subset $B \subset V$ is a basis for $V$ if $B$ is linearly independent and $\text{Span } B = V$.

**Theorem 2.4.1.** Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for $V$. Then
$$\forall \vec{v} \in V, \exists! a_1, \ldots, a_n \in F, \vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

**Proof.** Let $\vec{v} \in V$. Since $\text{Span } B = V$,
$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$
for some $a_1, \ldots, a_n \in F$. For uniqueness, let $b_1, \ldots, b_n \in F$ be such that
$$\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n.$$

Then
$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$$
$$(a_1 - b_1)\vec{v}_1 + \cdots + (a_n - b_n)\vec{v}_n = \vec{0}.$$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, we have
$$a_1 - b_1 = \cdots = a_n - b_n = 0,$$
so $a_i = b_i$ for all $i \in \{1, \ldots, n\}$. \qed

**Lemma 2.4.2.** If $S$ is a linearly independent subset of $V$, then $\forall \vec{u} \in V \smallsetminus \text{Span } S, S \cup \{\vec{u}\}$ is linearly independent.

**Proof.** If $V = \text{Span } S$, then $V \smallsetminus S = \emptyset$ and the statement is vacuously true. Assume that $\text{Span } S \subset V$ and let $\vec{u} \in V \smallsetminus \text{Span } S$.

We shall show that $S \cup \{\vec{u}\}$ is linearly independent.

Let $c_1, \ldots, c_n \in F$ and $\vec{u}_1, \ldots, \vec{u}_n \in S$ be such that
$$c\vec{u} + c_1\vec{u}_1 + \cdots + c_n\vec{u}_n = \vec{0}.$$

If $c \neq 0$, then
$$\vec{u} = -\frac{1}{c}(c_1\vec{u}_1 + \cdots + c_n\vec{u}_n)$$
is in $\text{Span } S$, which is a contradiction. Thus, $c = 0$, so
$$c_1\vec{u}_1 + \cdots + c_n\vec{u}_n = \vec{0}.$$

But $S$ is linearly independent, we get $c_1 = \cdots = c_n = 0$. \qed
Theorem 2.4.3. Let $V$ be a vector space over $F$.

(1) If $B$ is a linearly independent subset of $V$ which is maximal with respect to the property of being linearly independent (i.e., $\forall B \subseteq S, S \neq B \Rightarrow S$ is not linearly independent), then $B$ is a basis of $V$.

(2) If $B$ is a spanning set for $V$ which is minimal with respect to the property of spanning (i.e., $\forall S \subseteq B, S \neq B \Rightarrow \text{Span} \ S \not\subseteq V$), then $B$ is a basis of $V$.

Proof. (1) Assume that $B$ is a maximal linearly independent subset of $V$.

It remains to prove that $\text{Span} \ B = V$. Let $\vec{v} \in V \setminus \text{Span} \ B$.

By Lemma 2.4.2, $B \cup \{\vec{v}\}$ is linearly independent which contradicts the maximality of $B$. Hence, $\text{Span} \ B = V$.

(2) Let $B$ be a minimal spanning set for $V$.

We shall prove that $B$ is linearly independent.

Let $c_1, \ldots, c_n \in F$ and $\vec{v}_1, \ldots, \vec{v}_n \in B$ be such that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}.$$ 

Assume that $c_i \neq 0$ for some $i$. Then

$$\vec{v}_i = -\frac{1}{c_i} (c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n) \in \text{Span}(B \setminus \{v_i\}).$$

Thus, $V = \text{Span} \ B = \text{Span}(B \setminus \{v_i\})$ which contradicts the minimality of $B$.

Hence, $B$ is linearly independent. 

\[ \square \]

Theorem 2.4.4. Let $V$ be a vector space over $F$. If $V$ has a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, then any linearly independent set in $V$ has $\leq m$ elements.

Proof. Assume that $\text{Span} \{\vec{v}_1, \ldots, \vec{v}_m\} = V$.

Let $S$ be a linearly independent subset of $V$. Assume that $|S| > m$.

Let $T = \{\vec{u}_1, \ldots, \vec{u}_{m+1}\} \subseteq S \subseteq V$.

Since $\text{Span} \{\vec{v}_1, \ldots, \vec{v}_m\} = V$, we have

$$\vec{u}_1 = c_{11}\vec{v}_1 + c_{21}\vec{v}_2 + \cdots + c_{m1}\vec{v}_m$$
$$\vec{u}_2 = c_{12}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{m2}\vec{v}_m$$
$$\vdots$$
$$\vec{u}_{m+1} = c_{1m+1}\vec{v}_1 + c_{2m+1}\vec{v}_2 + \cdots + c_{m,m+1}\vec{v}_m$$

for some $c_{ij} \in F$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m+1\}$.

By Theorem 1.2.7, the matrix equation

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1,m+1} \\ c_{21} & c_{22} & \cdots & c_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{m,m+1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \end{bmatrix} = \vec{0}_m$$

has a nontrivial solution $\vec{x} \neq \vec{0}_{m+1}$. Note that

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_{m+1}\vec{u}_{m+1}$$
$$= x_1(c_{11}\vec{v}_1 + c_{21}\vec{v}_2 + \cdots + c_{m1}\vec{v}_m) + x_2(c_{12}\vec{v}_1 + c_{22}\vec{v}_2 + \cdots + c_{m2}\vec{v}_m)$$
$$+ \cdots + x_{m+1}(c_{1,m+1}\vec{v}_1 + c_{2,m+1}\vec{v}_2 + \cdots + c_{m,m+1}\vec{v}_m)$$
$$= (x_1c_{11} + x_2c_{12} + \cdots + x_{m+1}c_{1,m+1})\vec{v}_1 + (x_1c_{21} + x_2c_{22} + \cdots + x_{m+1}c_{2,m+1})\vec{v}_2$$
$$+ \cdots + (x_1c_{m1} + x_2c_{m2} + \cdots + x_{m+1}c_{m,m+1})\vec{v}_m$$
$$= \vec{0}$$

which contradicts $S$ is linearly independent. Hence, $|S| \leq m$. 

\[ \square \]
Corollary 2.4.5. If the vector space $V$ has a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_m\}$, then

1. $\{\vec{v}_1, \ldots, \vec{v}_m\}$ has a subset which is a basis for $V$,
2. any linearly independent set in $V$ can be extended to a basis for $V$,
3. $V$ has a basis,
4. any two bases for $V$ have the same finite number of elements, necessarily $\leq m$.

In this case, we say that $V$ is finite-dimensional, and the number of elements in a basis is called the dimension of $V$, written $\dim V$. If $V$ has no finite spanning set, we say that $V$ is infinite-dimensional.

Proof. (1) Let $\mathcal{S} = \{S \subseteq \{\vec{v}_1, \ldots, \vec{v}_m\} : \text{Span} = V\}$. Since $\{\vec{v}_1, \ldots, \vec{v}_m\} \in \mathcal{S}$, $\mathcal{S}$ is a nonempty finite set partially ordered by $\subseteq$. Then $\mathcal{S}$ has a minimal element, say $B$.

By Theorem 2.4.3 (2), $B$ is a basis for $V$.

(2) Let $A$ be a linearly independent subset of $V$.

If $\text{Span} A = V$, then $A$ is a basis for $V$. If not, let $\vec{v} \in V \setminus \text{Span} A$ and $A_1 = A \cup \{\vec{v}\}$. By Lemma 2.4.2, $A_1$ is linearly independent.

If $\text{Span} A_1 = V$, then $A_1$ is a basis for $V$. If not the case, we continue the same process, we get $A_k$ which is linearly independent.

Theorem 2.4.4 assures that the process must stop at $A_k$, where $|A_k| \leq m$ and $A_k$ is a basis for $V$.

(3) Since $\emptyset$ is linearly independent, $\emptyset$ can be extended to a basis for $V$ by (2). Hence, $V$ has a basis.

(4) Assume that $B_1$ and $B_2$ are bases for $V$. By Theorem 2.4.4, $|B_1|$ and $|B_2|$ are finite. Since $\text{Span} B_1 = V$ and $B_2$ is linearly independent, we have $|B_2| \leq |B_1|$ by Theorem 2.4.4. Interchange the roles of $B_1$ and $B_2$, we get $|B_1| \leq |B_2|$. Hence, $|B_1| = |B_2|$. \qed

Remark. The above proof is valid for a “finite” dimensional vector space. For a general (finite/infinite dimensional) vector space $V$, consider

$\mathcal{S} = \{S \subseteq V : S$ is linearly independent$\}$.

Then $\emptyset \in \mathcal{S}$. Partially order $\mathcal{S}$ by $\subseteq$. We can show that every chain in $\mathcal{S}$ has an upper bound. By Zorn’s lemma, $\mathcal{S}$ has a maximal element, say $B$. By Theorem 2.4.3 (1), $B$ is a basis of $V$. Hence, every vector space has a basis.
Example 2.4.1. Extend \{(1, 1, 1)\} to a basis of \(\mathbb{R}^3\).

Remark. If \(\vec{e}_i\) is the \(i\)th column of the identity matrix \(I_m\), then \(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}\) is a basis for \(F^m\), called the standard basis, and so \(\dim F^m = m\).

Corollary 2.4.6. If \(V\) is a vector space with \(\dim V = n\), then:
1. any spanning set of \(n\) elements is a basis of \(V\)
2. any linearly independent set of \(n\) elements is a basis of \(V\)
3. if \(W\) is an \(n\)-dimensional subspace of \(V\), then \(W = V\).

Proof. Let \(S = \{\vec{v}_1, \ldots, \vec{v}_n\}\) be a set of \(n\) vectors in \(V\).
1. Assume that \(\text{Span} \, S = V\). Then \(S\) has a subset which is a basis for \(V\) by Corollary 2.4.5 (1). But \(\dim V = n\), so we have \(S\) is a basis for \(V\).
2. Assume that \(S\) is linearly independent. Then \(S\) can be extended to a basis for \(V\) by Corollary 2.4.5 (2). Again \(\dim V = n\) implies \(S\) is a basis for \(V\).
3. Assume that \(W\) is a subspace of \(V\) and \(\dim W = n\) with basis \(B = \{\vec{w}_1, \ldots, \vec{w}_n\}\). Then \(B \subseteq V\) and \(B\) is linearly independent.
   But \(|B| = n = \dim V\), by (2), \(B\) is a basis for \(V\). Hence, \(W = \text{Span} \, B = V\). \(\square\)

Corollary 2.4.7. If \(V\) is a finite-dimensional vector space and \(U\) is a proper subspace of \(V\), then \(U\) is finite-dimensional and \(\dim U < \dim V\).

Proof. Assume that \(U\) is a proper subspace of \(V\).
Let \(B\) be a basis of \(U\). Then \(B\) is linearly independent.
By Corollary 2.4.5 (2), \(B\) can be extended to a basis \(C\) for \(V\).
If \(B = C\), then \(U = V\) by Corollary 2.4.6.
Hence, \(|B| < |C|\), so \(\dim U < \dim V\). \(\square\)
Theorem 2.4.8. If $W_1$ and $W_2$ are finite dimensional subspaces of a vector space $V$ over a field $F$, then $W_1 + W_2$ is finite dimensional and

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

**Proof.** Let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_r\}$ be a basis for $W_1 \cap W_2$.

Extend $\mathcal{B}$ to a basis $\mathcal{B}_1 = \{\vec{u}_1, \ldots, \vec{u}_r, \vec{v}_1, \ldots, \vec{v}_n\}$ for $W_1$ and to a basis $\mathcal{B}_2 = \{\vec{u}_1, \ldots, \vec{u}_r, \vec{w}_1, \ldots, \vec{w}_m\}$ for $W_2$.

Then $\dim W_1 = r + n$, $\dim W_2 = r + m$ and $\dim(W_1 \cap W_2) = r$.

We shall show that $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2 = \{\vec{u}_1, \ldots, \vec{u}_r, \vec{v}_1, \ldots, \vec{v}_n, \vec{w}_1, \ldots, \vec{w}_m\}$ is a basis for $W_1 + W_2$. This implies that

$$\dim(W_1 + W_2) = r + n + m = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

as desired.

By Exercise Set II · 8, we have

$$\text{Span}(\mathcal{B}_1 \cup \mathcal{B}_2) = \text{Span} \mathcal{B}_1 + \text{Span} \mathcal{B}_2 = W_1 + W_2.$$

Next, we prove linearly independence.

Let $c_1, \ldots, c_r, d_1, \ldots, d_n, e_1, \ldots, e_m \in F$ be such that

$$c_1 \vec{v}_1 + \cdots + c_r \vec{v}_r + d_1 \vec{v}_1 + \cdots + d_n \vec{v}_n + e_1 \vec{w}_1 + \cdots + e_m \vec{w}_m = \vec{0}.$$

Then

$$c_1 \vec{v}_1 + \cdots + c_r \vec{v}_r + d_1 \vec{v}_1 + \cdots + d_n \vec{v}_n = -e_1 \vec{w}_1 - \cdots - e_m \vec{w}_m.$$

so $-e_1 \vec{w}_1 - \cdots - e_m \vec{w}_m \in W_1 \cap W_2$.

Since $W_1 \cap W_2 = \text{Span}\{\vec{u}_1, \ldots, \vec{u}_r\}$, there exist $c'_1, \ldots, c'_r \in F$ such that

$$-e_1 \vec{w}_1 - \cdots - e_m \vec{w}_m = c'_1 \vec{u}_1 + \cdots + c'_r \vec{u}_r.$$

Thus,

$$(c_1 - c'_1) \vec{v}_1 + \cdots + (c_r - c'_r) \vec{v}_r + d_1 \vec{v}_1 + \cdots + d_n \vec{v}_n = \vec{0}.$$

Since $\mathcal{B}_1$ is linearly independent, $c_i = c'_i$ for all $i = 1, \ldots, r$ and $d_j = 0$ for all $j \in \{1, \ldots, n\}$. Therefore,

$$c_1 \vec{v}_1 + \cdots + c_r \vec{v}_r + e_1 \vec{w}_1 + \cdots + e_m \vec{w}_m = \vec{0}.$$

Since $\mathcal{B}_2$ is linearly independent, $c_i = 0$ for all $i = 1, \ldots, r$ and $e_k = 0$ for all $k \in \{1, \ldots, m\}$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent. \qed
Example 2.4.2. Consider two subspaces of $\mathbb{R}^5$

$$W_1 = \left\{ \begin{bmatrix} a \\ a-b \\ b \\ a+b \\ 0 \end{bmatrix} \in \mathbb{R}^5 : a, b \in \mathbb{R} \right\}$$

and

$$W_2 = \left\{ \begin{bmatrix} c \\ d \\ 0 \\ e \\ d-e \end{bmatrix} \in \mathbb{R}^5 : c, d, e \in \mathbb{R} \right\}.$$

Find bases for $W_1$, $W_2$ and $W_1 \cap W_2$. Determine the dimension of $W_1 + W_2$. 

Theorem 2.4.9. [Universal Mapping Property]

Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of $V$.

If $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$ (not necessarily distinct), then there exists a unique linear transformation $T : V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$.

Proof. By Theorem 2.4.1, for each $\vec{v} \in V$ can be expressed uniquely in the form

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n,$$

for some $c_1, c_2, \ldots, c_n \in F$, so if $T : V \to W$ is to be linear, we must define

$$T(\vec{v}) = T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n)$$

for all $\vec{v} \in V$.

Then $T$ is linear and $T(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$.

Finally, if $U : V \to W$ is another linear transformation such that $U(\vec{v}_i) = \vec{w}_i$ for all $i \in \{1, 2, \ldots, n\}$, then

$$U(\vec{v}) = U(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n) = c_1 U(\vec{v}_1) + c_2 U(\vec{v}_2) + \cdots + c_n U(\vec{v}_n)$$

because $U$ is linear,

$$= c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_n \vec{w}_n$$

$$= T(\vec{v})$$

for all $\vec{v} \in V$, and hence $U = T$. \qed
Example 2.4.3. (1) Find a linear transformation \( T \) that satisfies the following conditions
(a) \( T : \mathbb{C} \rightarrow \mathbb{R}^2[x] \) with \( T(1-i) = 2x^2 \) and \( T(1+i) = 1-x \),
(b) \( T : \mathbb{R}^2[x] \rightarrow \mathbb{R}^2 \) with
\[
T(1) = (0, 1), \quad T(1-x) = (0, 1) \quad \text{and} \quad T(x+x^2) = (1, 1).
\]

(2) Let \( T : \mathbb{R}_1[x] \rightarrow \mathbb{R}^3 \) be a linear transformation with
\[
T(2-x) = (1, -1, 1) \quad \text{and} \quad T(1+x) = (0, 1, -1).
\]
Find \( T(-1+2x) \).

Let \( V \) and \( W \) be vector spaces over a field and \( T : V \rightarrow W \) a linear transformation. The rank of \( T \), denoted by \( \text{rank} \ T \), is \( \dim(\text{im} \ T) \) and the nullity of \( T \), denoted by \( \text{nullity} \ T \), is \( \dim(\ker \ T) \).

Theorem 2.4.10. Let \( V \) and \( W \) be vector spaces over a field \( F \) and \( T : V \rightarrow W \) a linear transformation. If \( V \) is finite dimensional, then
\[
\text{rank} \ T + \text{nullity} \ T = \dim V.
\]

Proof. Let \( B = \{\vec{u}_1, \ldots, \vec{u}_k\} \) be a basis for \( \ker T \).
Extend \( B \) to a basis \( B' = \{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_m\} \) for \( V \). Then \( T(B') = \{\vec{0}_W, T(\vec{v}_1), \ldots, T(\vec{v}_m)\} \).
By Exercise Set II - 19, we have
\[
\text{im} \ T = \text{Span} \ T(B') = \text{Span} \{T(\vec{v}_1), \ldots, T(\vec{v}_m)\}.
\]
We shall prove that the set \( \{T(\vec{v}_1), \ldots, T(\vec{v}_m)\} \) is linear independent and hence it is a basis for \( \text{im} \ T \). Let \( c_1, \ldots, c_m \in F \) be such that
\[
c_1T(\vec{v}_1) + \cdots + c_mT(\vec{v}_m) = \vec{0}_m.
\]
Then
\[
T(c_1\vec{v}_1 + \cdots + c_m\vec{v}_m) = \vec{0}_m,
\]
so \( c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in \ker T \). Thus,
\[
c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = d_1\vec{u}_1 + \cdots + d_k\vec{u}_k
\]
for some \( d_1, \ldots, d_k \in F \), so
\[
c_1\vec{v}_1 + \cdots + c_m\vec{v}_m + (-d_1)\vec{u}_1 + \cdots + (-d_k)\vec{u}_k = \vec{0}_V.
\]
Since \( B' \) is linearly independent, we get
\[
c_1 = \cdots = c_m = d_1 = \cdots = d_k = 0.
\]
Hence, \( \{T(\vec{v}_1), \ldots, T(\vec{v}_m)\} \) is linear independent. Therefore,
\[
\text{rank} \ T + \text{nullity} \ T = \dim(\text{im} \ T) + \dim(\ker \ T) = m + k = \dim V
\]
as desired. \( \square \)
**Theorem 2.4.11.** Let $V$ and $W$ be finite dimensional and $T : V \to W$ a linear transformation and $\dim V = \dim W$. Then $T$ is 1-1 $\iff$ $T$ is onto.

**Proof.** Let $T$ be a linear transformation.

\[ T \text{ is 1-1 } \iff \ker T = \{ \vec{0}_V \} \]
\[ \iff \text{nullity } T = 0 \]
\[ \iff \rank T = \dim V \]
\[ \iff \rank T = \dim W \]
\[ \iff \dim(\im T) = \dim W \]
\[ \iff \im T = W \]
\[ \iff T \text{ is onto.} \]

Hence, $T$ is 1-1 $\iff$ $T$ is onto. \hfill \square

**Corollary 2.4.12.** If $V$ is finite dimensional, $S$ and $T$ are linear transformations from $V$ to $V$, and $T \circ S$ is the identity map, then $T = S^{-1}$.

**Proof.** We shall prove that $S$ is 1-1. Let $\vec{x} \in V$ be such that $S(\vec{x}) = \vec{0}_V$.

Since $T \circ S$ is the identity map, we have

\[ \vec{x} = (T \circ S)(\vec{x}) = T(\vec{0}_V) = \vec{0}_V. \]

By Theorem 2.3.5, $S$ is 1-1. Thus, $S$ is onto by Theorem 2.4.11.

Hence, $S$ is invertible and $T = S^{-1}$. \hfill \square

From Theorem 2.4.1, we know that the representation of a given vector $\vec{v} \in V$ in terms of a given basis is unique. Let $V$ be an $n$-dimensional vector space over a field $F$ with an ordered basis $B = \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \}$ and $\vec{v} \in V$.

Then

\[ \forall \vec{v} \in V, \exists! (c_1, \ldots, c_n) \in F^n, \vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_n \vec{u}_n. \]

The vector $[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$ is called the *coordinate vector of $\vec{v}$ relative to the ordered basis $B$*.

A one-to-one linear transformation from $V$ onto $W$ is called an isomorphism. If there exists an isomorphism from $V$ onto $W$, then we say that $V$ is isomorphic to $W$ and we write $V \cong W$. Note that $\cong$ is an equivalence relation.

**Theorem 2.4.13.** If $V$ is an $n$-dimensional vector space over $F$, then $V \cong F^n$.

**Proof.** Let $B = \{ \vec{u}_1, \ldots, \vec{u}_n \}$ be an ordered basis for $V$.

Define $T : V \to F^n$ by

\[ \vec{v} = c_1 \vec{u}_1 + \cdots + c_n \vec{u}_n \mapsto \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\vec{v}]_B. \]

Let $\vec{v}, \vec{w} \in V$ and $a \in F$.

Then $\vec{v} = c_1 \vec{u}_1 + \cdots + c_n \vec{u}_n$ and $\vec{w} = d_1 \vec{u}_1 + \cdots + d_n \vec{u}_n$, so

\[ [\vec{v} + \vec{w}]_B = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\vec{v}]_B + [\vec{w}]_B \]

and

\[ [a \vec{v}]_B = \begin{bmatrix} ac_1 \\ \vdots \\ ac_n \end{bmatrix} = a \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = a[\vec{v}]_B. \]

Hence, $T$ is a linear transformation.
Next, we shall show that \( T \) is onto. Let \( \vec{v} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^n \). Choose \( \vec{v} = c_1 \vec{u}_1 + \cdots + c_n \vec{u}_n \in V \). Then \( T(\vec{v}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \).

Since \( \dim V = n = \dim \mathbb{F}^n \), we have \( T \) is 1-1 by 2.4.11. Therefore, \( T \) is an isomorphism and \( V \cong \mathbb{F}^n \).

**Corollary 2.4.14.** If \( V \) and \( W \) are finite dimensional, then

\[
\dim V = \dim W \iff V \cong W.
\]

Let \( A \) be an \( m \times n \) matrix. When \( A \) is row reduced to a matrix \( B \), the column of \( B \) are often totally different from the columns of \( A \). However, the equation \( A\vec{x} = \vec{0} \) and \( B\vec{x} = \vec{0} \) have exactly the same set of solutions. That is, the column of \( A \) have exactly the same linear dependence relationships as the column of \( B \). We conclude this result in

**Theorem 2.4.15.** Elementary row operations on a matrix do not effect the linear dependence relations among the columns of the matrix. Moreover, the pivot columns of a matrix form a basis for \( \text{Col} A \). Hence, \( \dim(\text{Col} A) = \text{rank} A \).

For an \( m \times n \) matrix \( A \), \( \dim(\text{Nul} A) \), denoted by \( \text{nullity} A \), is called the **nullity of \( A \)**.

Recall that \( T : \vec{x} \to A\vec{x} \) is a linear transformation from \( \mathbb{F}^n \) to \( \mathbb{F}^m \).

**Corollary 2.4.16.** Let \( A \) be an \( m \times n \) matrix. Then

\[
\text{nullity} A + \text{rank} A = n.
\]

**Proof.** It follows from Theorem 2.4.10. \( \square \)
Theorem 2.4.17. Each $m \times n$ matrix $A$ is row equivalent to a unique reduced echelon matrix $U$.

Proof. The row reduction algorithm shows that there exists at least one such matrix $U$. Suppose that $A$ is row equivalent to matrices $U$ and $V$ in reduced echelon form. The leftmost nonzero entry in a row of $U$ is a “leading 1”.

The pivot columns of $U$ and $V$ are precisely the nonzero columns that are not linearly dependent on the columns to their left. (This condition is satisfied automatically by a first column if it is nonzero.) Since $U$ and $V$ are row equivalent (both being row equivalent to $A$), their columns have the same linear dependence relations. Hence, the pivot columns of $U$ and $V$ appear in the same locations. If there are $r$ such columns, then since $U$ and $V$ are in reduced echelon form, their pivot columns are the first $r$ columns of the $m \times m$ identity matrix. Thus, corresponding pivot columns of $U$ and $V$ are equal.

Finally, consider any nonpivot column of $U$, say column $j$. This column is either zero or a linear combination of the pivot columns to its left (because those pivot columns are a basis for the space spanned by the columns to the left of column $j$). Either case can be expressed by writing $U\vec{x} = \vec{0}$ for some $\vec{x}$ whose $j$th entry is 1. Then $V\vec{x} = \vec{0}$, too, which says that column $j$ of $V$ is either zero or the same linear combination of the pivot columns of $V$ to its left. Since corresponding pivot columns of $U$ and $V$ are equal, columns $j$ of $U$ and $V$ are also equal. This holds for all nonpivot columns, so $V = U$, which proves that $U$ is unique. \qed