Chapter 3

Number-Theoretic Functions

3.1 Multiplicative Functions

Definition. A real- or complex-valued function defined on the positive integers is called an arithmetic function or a number-theoretic function.

Throughout this chapter, variables occurring as arguments of number-theoretic functions are understood to be positive. The same applies to their divisors.

Examples 3.1.1. The following functions are arithmetic functions.

1. $\phi(n) = |\{r \in \mathbb{Z} : 0 \leq r < n \text{ and } \gcd(r, n) = 1\}|$.
2. $\tau(n)$ is the number of positive divisors of $n = \sum_{d|n} 1$.
3. $\sigma(n)$ is the sum of positive divisors of $n = \sum_{d|n} d$.

Here $\sum_{d|n} f(d)$ means the sum of the values $f(d)$ as $d$ runs over all positive divisors of the positive integer $n$. E.g., $\sum_{d|12} f(d) = f(1) + f(2) + f(3) + f(4) + f(6) + f(12)$.

Theorem 3.1.1. Let $p$ be a prime and $k \in \mathbb{N} \cup \{0\}$. Then

$$\tau(p^k) = |\{1, p, p^2, \ldots, p^k\}| = k + 1$$

and

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$
Definition. A number-theoretic function $f$ which is not identically zero is said to be **multiplicative** if $\forall m,n \in \mathbb{N}, \gcd(m,n) = 1 \Rightarrow f(mn) = f(m)f(n)$.

**Example 3.1.2.** The following functions are multiplicative.

(i) $\phi$ (Theorem 2.3.6)  
(ii) $U(n) = 1$ for all $n \in \mathbb{N}$  
(iii) $N(n) = n$ for all $n \in \mathbb{N}$.

**Remark.** Let $f$ be a multiplicative function. Then $f(1) = f(1 \cdot 1) = f(1)f(1)$, so $f(1) = 0$ or $1$. If $f(1) = 0$, then $f(n) = f(1 \cdot n) = f(1)f(n) = 0$, so $f$ is the zero function. Hence, if $f$ is multiplicative, then $f(1) = 1$.

**Lemma 3.1.2.** $f$ is multiplicative $\iff f(1) = 1$ and $f(p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{r}^{k_{r}}) = f(p_{1}^{k_{1}}) f(p_{2}^{k_{2}}) \ldots f(p_{r}^{k_{r}})$ for all distinct primes $p_{i}$ and $r, k_{i} \in \mathbb{N}$.

**Remarks.**

1. From the above lemma, to compute the values of a multiplicative function $f$, it suffices to know only the values of $f(p^{k})$ for all primes $p$ and $k \in \mathbb{N}$.

2. If $f$ and $g$ are multiplicative functions and $f(p^{k}) = g(p^{k})$ for all primes $p$ and $k \in \mathbb{N}$, then $f = g$.

**Definition.** A number-theoretic function $f$ which is not identically zero is said to be **completely multiplicative** if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

E.g., (i) $U(n) = 1$, for all $n \in \mathbb{N}$, and (ii) $N(n) = n$, for all $n \in \mathbb{N}$, are completely multiplicative.

**Remark.** If $f$ is completely multiplicative, then

$$f(p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{r}^{k_{r}}) = f(p_{1})^{k_{1}} f(p_{2})^{k_{2}} \ldots f(p_{r})^{k_{r}}.$$ 

Thus, to determine the values of a completely multiplicative function $f$, it suffices to know only the values of $f(p)$ for all primes $p$.

**Lemma 3.1.3.** Let $n > 1$ be factored as $n = p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{r}^{k_{r}}$ for some primes $p_{i}$ and $r, k_{i} \in \mathbb{N}$ for all $i \in \{1, 2, \ldots, r\}$. Then for $d \in \mathbb{N}$,

$$d \mid n \iff d = p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots p_{r}^{a_{r}}, \text{ where } 0 \leq a_{i} \leq k_{i} \text{ for all } i \in \{1, 2, \ldots, r\}.$$ 

Hence, $\{d \in \mathbb{N} : d \mid n\} = \{p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots p_{r}^{a_{r}} : 0 \leq a_{i} \leq k_{i} \text{ for all } i \in \{1, 2, \ldots, r\}\}$.

**Theorem 3.1.4.** If $n = p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{r}^{k_{r}}$ is the prime factorization of $n > 1$, then

$$\tau(n) = (k_{1} + 1)(k_{2} + 2) \ldots (k_{r} + 1) = \tau(p_{1}^{k_{1}}) \tau(p_{2}^{k_{2}}) \ldots \tau(p_{r}^{k_{r}}).$$ 

Moreover, $\tau$ is multiplicative.
Definition. A positive integer \( n \) is a perfect square number if \( \exists a \in \mathbb{Z}, n = a^2 \).

Remarks. 1. If \( n \) is a perfect square number, then \( n \equiv 0 \text{ or } 1 \pmod{4} \).

2. \( n \) is a perfect square if and only if \( \tau(n) \) is odd.

Let \( n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \) is the prime factorization of \( n > 1 \). Consider the product
\[
(1 + p_1 + p_1^2 + \cdots + p_1^{k_1})(1 + p_2 + p_2^2 + \cdots + p_2^{k_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{k_r})
\]
\[
= \sum_{d \mid n} \sigma(d)
\]
\[
= \sum_{a_i \leq k_i} q_i^{b_i}
\]
\[
= \sigma(p_1^{k_1}) \sigma(p_2^{k_2}) \cdots \sigma(p_r^{k_r}).
\]

Moreover, \( \sigma \) is multiplicative.

Lemma 3.1.6. Assume that \( \gcd(m, n) = 1 \). Then
\[
|d \mid n : d \mid mn| = |d_1 d_2 : d_1, d_2 \in \mathbb{N}, d_1 \mid m, d_2 \mid n \text{ and } \gcd(d_1, d_2) = 1|.
\]

Proof. The result is clear when \( m \) or \( n \) is 1. Assume that \( m, n > 1 \) and \( \gcd(m, n) = 1 \).

Let \( m = p_1^{m_1} \cdots p_r^{m_r} \) and \( n = q_1^{n_1} \cdots q_s^{n_s} \), where \( p_i \) and \( q_i \) are all distinct primes and \( m_i, n_j \in \mathbb{N} \) for all \( i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, s\} \).

Suppose that \( d \mid mn \). By Lemma 3.1.3, \( d = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} \) for some \( 0 \leq a_i \leq m_i \) and \( 0 \leq b_j \leq n_j \) for all \( i, j \). Thus \( d = d_1 d_2 \) where \( d_1 = p_1^{a_1} \cdots p_r^{a_r} \), \( d_2 = q_1^{b_1} \cdots q_s^{b_s} \), so \( d_1 \mid m, d_2 \mid n \) and \( \gcd(d_1, d_2) = 1 \). The converse is clear. \( \square \)

Remark. If \( \gcd(m, n) = 1 \), then the above lemma gives
\[
\sum_{d \mid mn} f(d) = \sum_{d_1 d_2, \gcd(d_1, d_2) = 1} f(d_1 d_2).
\]

Theorem 3.1.7. If \( f \) is multiplicative function and \( F \) is defined by
\[
F(n) = \sum_{d \mid n} f(d),
\]
then \( F \) is also multiplicative.
Proof. Let \( m, n \in \mathbb{N} \) be such that \( \gcd(m, n) = 1 \). Then

\[
F(mn) = \sum_{d \mid mn} f(d) = \sum_{d \mid m, d \mid n, \gcd(d_1, d_2) = 1} f(d_1) f(d_2) = \sum_{d_1 \mid m, d_2 \mid n} f(d_1) f(d_2) \quad \text{(since } \gcd(d_1, d_2) = 1) \nonumber
\]

\[
= \sum_{d_1 \mid m, d_2 \mid n} f(d_1) \sum_{d_2 \mid n} f(d_2) = F(m)F(n). \nonumber
\]

Hence, \( F \) is multiplicative. \( \square \)

Recall that \( U(n) = 1 \) for all \( n \in \mathbb{N} \) and \( N(n) = n \) for all \( n \in \mathbb{N} \) are multiplicative. The above theorem gives another proof of the following result.

Corollary 3.1.8. \( \tau(n) = \sum_{d \mid n} 1 \) and \( \sigma(n) = \sum_{d \mid n} d \) are multiplicative.

Theorem 3.1.9. \( \sum_{d \mid n} \phi(d) = n \)

Proof. We first observe that

\[
\left\{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} \right\} = \bigcup_{d \mid n} \left\{ \frac{a}{d} : 1 \leq a \leq d \text{ and } \gcd(a, d) = 1 \right\}. \nonumber
\]

Moreover, for \( d \mid n \), each set in the union is of cardinality \( \phi(d) \). Assume that \( d_1 \mid n \), \( d_2 \mid n \) and \( \frac{a}{d_1} = \frac{b}{d_2} \) for some \( 1 \leq a \leq d_1, \gcd(a, d_1) = 1 \) and \( 1 \leq b \leq d_2, \gcd(b, d_2) = 1 \). Then \( ad_2 = bd_1 \) which implies \( d_1 \mid ad_2 \) and \( d_2 \mid bd_1 \). Since \( \gcd(a, d_1) = 1 = \gcd(b, d_2) \), \( d_1 \mid d_2 \) and \( d_2 \mid d_1 \) by Corollary 1.1.12, so \( d_1 = d_2 \) and \( a = b \). This shows that the union on the right hand side is a disjoint union. Hence,

\[
n = \left| \left\{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} \right\} \right| = \left| \bigcup_{d \mid n} \left\{ \frac{a}{d} : 1 \leq a \leq d \text{ and } \gcd(a, d) = 1 \right\} \right| = \sum_{d \mid n} \left| \left\{ \frac{a}{d} : 1 \leq a \leq d \text{ and } \gcd(a, d) = 1 \right\} \right| = \sum_{d \mid n} \phi(d) \nonumber
\]

as desired. \( \square \)

Exercise 3.1. 1. Find the smallest \( n \in \mathbb{N} \) such that \( \tau(n) = 10 \).

2. Prove that \( \sum_{d \mid n} \tau^2(d) = \left( \sum_{d \mid n} \tau(d) \right)^2 \).

3. Prove that \( \sigma(n) \) is odd if and only if \( n \) is a perfect square or twice a perfect square.

4. Prove that \( \phi(m) \phi(n) = \phi(\gcd(m, n)) \phi(\operatorname{lcm}(m, n)) \) for all \( m, n \in \mathbb{N} \).

5. Show that the number of ordered pairs of positive integers whose lcm is \( n \) is \( \tau(n^2) \).

6. (i) For a fixed integer \( k \), show that the function \( f_k(n) = n^k \) for all \( n \in \mathbb{N} \) is multiplicative.

(ii) For each \( k \in \mathbb{N} \), show that the function \( \sigma_k(n) = \sum_{d \mid n} d^k \) for all \( n \in \mathbb{N} \) is multiplicative and find a formula for it.
3.2 The Möbius Inversion Formula

Definition. An integer $n$ is said to be square-free if it is not divisible by the square of any prime.

Remark. Every positive integer $n$ can be written uniquely in the form $n = ab^2$, where $a, b \in \mathbb{N}$ and $a$ is square-free.

Definition. [Möbius, 1832] For a positive integer $n$, we define the Möbius function, $\mu$, by the rules

$$
\mu(n) = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } \exists \text{ a prime } p, p^2 \mid n, \text{i.e., } n \text{ is not square-free}, \\
(-1)^r, & \text{if } n = p_1 p_2 \ldots p_r, \text{where } p_1, p_2, \ldots, p_r \text{ are distinct primes.}
\end{cases}
$$

Theorem 3.2.1. The Möbius function $\mu$ is multiplicative.

Proof. Note that $\mu(1) = 1$. Suppose $n > 1$ and write $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, where $p_i$ are distinct primes and $k_i \geq 1$ for all $i$. If $k_i > 1$ for some $j \in \{1, 2, \ldots, r\}$, we have $\mu(n) = 0$ and $\mu(p_j^{k_j}) = 0$, so $\mu(n) = \mu(p_1^{k_1})\mu(p_2^{k_2}) \ldots \mu(p_r^{k_r})$. Assume that $k_i = 1$ for all $i$. Then $n = p_1 p_2 \ldots p_r$, so $\mu(n) = (-1)^r$. Since $\mu(p_i) = -1$ for all $i$, we have $\mu(p_1)\mu(p_2) \cdot \mu(p_r) = (-1)^r = \mu(n)$. Hence, $\mu$ is multiplicative by Lemma 3.1.2. □

Theorem 3.2.2. $E(n) = \sum_{d \mid n} \mu(d) = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n > 1,
\end{cases}$ for all $n \in \mathbb{N}$ is a multiplicative function.

Proof. By Theorem 3.1.7, $E$ is multiplicative, and since

$$
E(p^k) = \begin{cases} 
1, & \text{if } k = 0, \\
1 - 1 + 0 + \cdots + 0, & \text{if } k \geq 1,
\end{cases}
$$

we see that $E(n) = 0$ if $n$ is divisible by a prime $p$, that is, if $n > 1$. □
Remark. For \( n \in \mathbb{N} \), \( \{d \in \mathbb{N} : d \mid n\} = \{n/d : d \in \mathbb{N} \text{ and } d \mid n\} \).

Lemma 3.2.3. Let \( f \) and \( g \) be multiplicative functions. Then

1. \( fg \) and \( f/g \) are multiplicative (whenever the latter function is defined), and
2. \( F(n) = \sum_{d \mid n} f(d)g(n/d) = \sum_{d \mid n} f(n/d)g(d) \) is a multiplicative function.

Proof. Exercises. □

Definition. For arithmetic functions \( f \) and \( g \), we define the Dirichlet convolution by

\[
(f \ast g)(n) = \sum_{d \mid n} f(d)g(n/d)
\]

for all \( n \in \mathbb{N} \).

Remarks. 1. Clearly, \( f \ast g = g \ast f \) and we can verify that \( f \ast (g \ast h) = (f \ast g) \ast h \).

2. By Lemma 3.2.3, if \( f \) and \( g \) are multiplicative, then \( f \ast g \) is also multiplicative.

3. The set of multiplicative functions is an abelian group under the Dirichlet convolution with identity element \( E(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \)

Theorem 3.2.4. [Möbius Inversion Formula] Let \( F \) and \( f \) be two arithmetic functions (not necessarily multiplicative) related by the formula

\[
F(n) = \sum_{d \mid n} f(d) = (f \ast \mu)(n).
\]

Then \( f(n) = \sum_{d \mid n} F(d)\mu(n/d) = \sum_{d \mid n} F(n/d)\mu(d) = \sum_{d_1d_2 = n} F(d_1)\mu(d_2) \), i.e., \( f = F \ast \mu \).

Proof. We have

\[
\sum_{d \mid n} F(n/d)\mu(d) = \sum_{d_1d_2 = n} F(d_1)\mu(d_2) = \sum_{d_1d_2 = n} \left( \sum_{\substack{d \mid n \mid (n/d)}} f(d) \right) \mu(d_2) = \sum_{d \mid n} f(d) \sum_{d_2 | (n/d)} \mu(d_2).
\]

But \( \sum_{d_2 | (n/d)} \mu(d_2) = 0 \) unless \( n/d = 1 \) (that is, unless \( d = n \)) when it is 1, so that this last sum is equal to \( f(n) \). □
Example 3.2.1. We know by Theorem 3.1.9 that $\sum_{d\mid n} \phi(d) = n$, i.e., $\phi \ast U = N$. The Möbius inversion formula gives $\phi = N \ast \mu$, i.e., we have

$$\phi(n) = \sum_{d\mid n} \frac{n}{d} \mu(d) = n \sum_{d\mid n} \frac{\mu(d)}{d} \quad \text{for all } n \in \mathbb{N}.$$

Corollary 3.2.5. Let $F$ and $f$ be two arithmetic functions related by the formula

$$F(n) = \sum_{d\mid n} f(d).$$

If $F$ is multiplicative, then $f$ is also multiplicative.

Proof. It follows from Theorem 3.2.4 and Theorems 3.2.3, 3.2.1.

Corollary 3.2.6. Let $F$ and $f$ be two arithmetic functions related by the formula

$$F(n) = \prod_{d\mid n} f(d).$$

Then $f(n) = \prod_{d\mid n} F(n/d) \mu(d)$.

Proof. Its proof is similar to the Möbius inversion formula and is left as an exercise.

Exercise 3.2.

1. Prove Lemma 3.2.3 and Corollary 3.2.6.

2. Prove that $\sum_{d\mid n} \sigma(d) \mu(n/d) = n$ for all $n \in \mathbb{N}$.

3. Let $f, g$ and $h$ be arithmetic functions. Prove that
   
   (i) $f \ast (g \ast h) = (f \ast g) \ast h$,
   (ii) $f \ast (g + h) = f \ast g + f \ast h$,
   (iii) $(\exists$ an arithmetic function $F$ such that $f \ast F = E$) if and only if $f(1) \neq 0$.

4. Determine the arithmetic function $f$ such that $\mu = f \ast U$. Is $f$ multiplicative? If so, find its values on the prime powers.

5. Show that if $f$ is multiplicative, then $\sum_{d\mid n} \mu(d)f(d) = \prod_{p \mid n} (1 - f(p))$.

6. Show that $\prod_{d\mid n} d = n^{\tau(n)/2}$ for all $n \in \mathbb{N}$.

3.3 The Greatest Integer Function

Let $x \in \mathbb{R}$. By Archimedean property and well-ordering principle, we can prove that there exists an $n_x \in \mathbb{Z}$ such that

$$n_x \leq x < n_x + 1.$$

This leads to the following definition.
Definition. For each real number \( x \), \( \lfloor x \rfloor \) is the unique integer such that
\[
x - 1 < \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.
\]
That is, \( \lfloor x \rfloor \) is the largest integer \( \leq x \). Sometimes, \( \lfloor x \rfloor \) is called the floor of \( x \). Note that \( \lfloor x \rfloor = \max((-\infty, x] \cap \mathbb{Z}) \). The greatest integer function is the map \( x \mapsto \lfloor x \rfloor \) for all \( x \in \mathbb{R} \).

Some properties of \( \lfloor x \rfloor \) are listed in the following theorem.

**Theorem 3.3.1.** Let \( x, x_1 \) and \( x_2 \) be real numbers.

1. \( x = \lfloor x \rfloor + \{x\} \), where \( 0 \leq \{x\} < 1 \). \( \{x\} \) is called the fractional part of \( x \).
2. \( \lfloor x \rfloor = x \) if and only if \( x \) is an integer.
3. \( \lfloor x + a \rfloor = \lfloor x \rfloor + a \), if \( a \in \mathbb{Z} \).
4. \( \lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ -1, & \text{otherwise}. \end{cases} \)
5. \( \lfloor x_1 \rfloor + \lfloor x_2 \rfloor \leq \lfloor x_1 + x_2 \rfloor \leq \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + 1. \)
6. \( \lfloor x/n \rfloor = \lfloor x \rfloor/n \) if \( n \in \mathbb{N} \).
7. \( -\lfloor -x \rfloor \) is the least integer \( \geq x \) and \( \lfloor x + 1/2 \rfloor \) is the nearest integer to \( x \).
8. \( 0 \leq \lfloor x \rfloor - 2\lfloor x/2 \rfloor \leq 1. \)
9. If \( x_1 < x_2 \), then \( |(x_1, x_2] \cap \mathbb{Z}| = [x_2] - [x_1]. \)
10. For \( d \in \mathbb{N} \) and \( x > 0 \), \( \{n \in \mathbb{N} : d \mid n \text{ and } n \leq x\} = [x/d] \), so \( \sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} [x/k] \).

**Theorem 3.3.2.** If \( a \in \mathbb{Z} \) and \( m \in \mathbb{N} \), then
\[
a = m\lfloor a/m \rfloor + m\{a/m\} \quad \text{and} \quad 0 \leq m\{a/m\} < m.
\]
That is, \( \lfloor a/m \rfloor \) and \( m\{a/m\} \) are the quotient and the remainder in the division of \( a \) by \( m \).

We write \( p^e \mid n \) if \( p^e \mid n \) and \( p^{e+1} \nmid n \), i.e., \( e \) is the highest exponent of \( p \) that divides \( n \).

**Theorem 3.3.3.** [de Polignac’s Formula] If \( n \) is a positive integer and \( p \) is a prime, then the highest exponent of \( p \) that divides \( n! \) is
\[
e = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots = \sum_{j=1}^{\infty} [n/p^j].
\]
That is, \( p^e \mid n! \).
Proof. The sum has only finitely many nonzero terms, since \([n/p^k] = 0\) for \(p^k > n\). Note that if \(p > n\), then \(p \nmid n!\) and \(\sum_{j=1}^{\infty} [n/p^j] = 0\). If \(p \leq n\), then \([n/p]\) integers in \([1, 2, \ldots, n]\) are divisible by \(p\), namely,
\[
p, 2p, 3p, \ldots, [n/p]p.
\]
Of these integers, \([n/p^2]\) are again divisible by \(p^2\):
\[
p^2, 2p^2, \ldots, [n/p^2]p^2.
\]
By the same idea, \([n/p^3]\) of these are divisible by \(p^3\):
\[
\]
After finitely many repetitions of this argument, the total number of times \(p\) divides number in \([1, 2, \ldots, n]\) is precisely
\[
\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.
\]
Hence, this sum is the exponent of \(p\) appearing in the prime factorization of \(n!\).

**Remark.** Recall that \([x/k] = \lfloor x/k \rfloor\) if \(k \in \mathbb{N}\), this shortens the computation for \(e\) as follows:
\[
e = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{[n/p]/p}{p} \right\rfloor + \cdots.
\]

**Example 3.3.1.** Find the highest power of 7 that divides 1000!.

**Proof.** We compute \(\left\lfloor \frac{1000}{7} \right\rfloor = 142\), \(\left\lfloor \frac{142}{7} \right\rfloor = 20\), \(\left\lfloor \frac{20}{7} \right\rfloor = 2\) and \(\left\lfloor \frac{2}{7} \right\rfloor = 0\). Thus \(e = 142 + 20 + 2 + 0 = 164\) is the highest power of 7 divides 1000!.

**Theorem 3.3.4.** If \(0 \leq k \leq n\), then the binomial coefficient \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\) is an integer.

**Proof.** This follows from the fact that
\[
[n/p^j] = [(n-k+k)/p^j] \geq [(n-k)/p^j] + [k/p^j]
\]
for all \(j \in \mathbb{N}\).

**Corollary 3.3.5.** For \(k \in \mathbb{N}\), \(k!\) divides the product of \(k\) consecutive integers.

**Exercise 3.3.**
1. Prove Theorem 3.3.1 (7)–(9).
2. Prove that if \(n \in \mathbb{N}\) and \(\alpha\) is a non-negative real number, then \(\sum_{j=1}^{n-1} \left\lfloor \alpha + \frac{k}{n} \right\rfloor = [\alpha n]\).
3. (i) Let $F$ and $f$ be two arithmetic functions related by the formula $F(n) = \sum_{d|n} f(d)$.

Prove that $\sum_{k=1}^{n} F(k) = \sum_{k=1}^{n} f(k)[n/k]$ for all $n \in \mathbb{N}$.

(ii) Conclude that $\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} [n/k]$ and $\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k[n/k]$.

(iii) Evaluate the sum $\sum_{k=1}^{n} [n/k] \phi(n)$.

4. Find the highest power of 17 that divides 2010!.

5. (i) Verify that 1000! terminates in 249 zeros.

(ii) For what values of $n$ does $n!$ terminate in 37 zeros.

6. Find the greatest common divisor of the binomial coefficients $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$. 