Least Integral-Squared Error Design of FIR Filters

- Let $H_d(e^{j\omega})$ denote the desired frequency response
- Since $H_d(e^{j\omega})$ is a periodic function of ω with a period 2π, it can be expressed as a Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

where

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \le n \le \infty$$

Least Integral-Squared Error Design of FIR Filters

- In general, $H_d(e^{j\omega})$ is piecewise constant with sharp transitions between bands
- In which case, $\{h_d[n]\}$ is of infinite length and noncausal
- Objective - Find a finite-duration $\{h_t[n]\}$ of length $2M+1$ whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$ in some sense

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Least Integral-Squared Error Design of FIR Filters

- Commonly used approximation criterion - Minimize the integral-squared error

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_t(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 d\omega$$

where

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n}$$

Least Integral-Squared Error Design of FIR Filters

- Using Parseval's relation we can write

$$\Phi = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

$$= \sum_{n=-M}^{M} \left| h_t[n] - h_d[n] \right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

- It follows from the above that Φ is minimum when $h_t[n] = h_d[n]$ for $-M \le n \le M$
- $\Rightarrow$ Best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncation

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Least Integral-Squared Error Design of FIR Filters

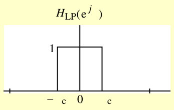
- A causal FIR filter with an impulse response $h[n]$ can be derived from $h_t[n]$ by delaying:

$$h[n] = h_t[n-M]$$

- The causal FIR filter $h[n]$ has the same magnitude response as $h_t[n]$ and its phase response has a linear phase shift of ωM radians with respect to that of $h_t[n]$

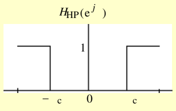
Impulse Responses of Ideal Filters

- Ideal lowpass filter -



$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$

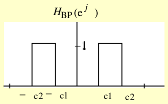
- Ideal highpass filter -



$$h_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

Impulse Responses of Ideal Filters

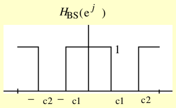
- Ideal bandpass filter -



$$h_{BP}[n]=\begin{cases}\dfrac{\sin(\omega_{c2}n)}{\pi n}-\dfrac{\sin(\omega_{c1}n)}{\pi n}, & n\neq 0\\[2ex] \dfrac{\omega_{c2}}{\pi}-\dfrac{\omega_{c1}}{\pi}, & n=0\end{cases}$$

Impulse Responses of Ideal Filters

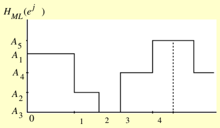
- Ideal bandstop filter -



$$h_{BS}[n]=\begin{cases}1-\dfrac{(\omega_{c2}-\omega_{c1})}{\pi}, & n=0\\[2ex] \dfrac{\sin(\omega_{c1}n)}{\pi n}-\dfrac{\sin(\omega_{c2}n)}{\pi n}, & n\neq 0\end{cases}$$

Impulse Responses of Ideal Filters

- Ideal multiband filter -



$$H_{ML}(e^{j\omega})=A_k,$$
$$\omega_{k-1}\leq\omega\leq\omega_k,$$
$$k=1,2,\ldots,L$$

$$h_{ML}[n]=\sum_{\ell=1}^{L}(A_\ell-A_{\ell+1})\cdot\frac{\sin(\omega_L n)}{\pi n}$$

Impulse Responses of Ideal Filters

- Ideal discrete-time Hilbert transformer -

$$H_{HT}(e^{j\omega})=\begin{cases}j, & -\pi<\omega<0\\ -j, & 0<\omega<\pi\end{cases}$$

$$h_{HT}[n]=\begin{cases}0, & \text{for } n \text{ even}\\ 2/\pi n, & \text{for } n \text{ odd}\end{cases}$$

Impulse Responses of Ideal Filters

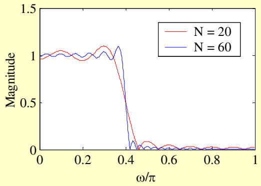
- Ideal discrete-time differentiator -

$$H_{DIF}(e^{j\omega})=j\omega,\quad 0\leq|\omega|\leq\pi$$

$$h_{DIF}[n]=\begin{cases}0, & n=0\\ \dfrac{\cos\pi n}{n}, & n\neq 0\end{cases}$$

Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Gibbs Phenomenon

- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths
- Height of the largest ripples remain the same independent of length
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters

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Gibbs Phenomenon

- Gibbs phenomenon can be explained by treating the truncation operation as an windowing operation:

$$h_t[n] = h_d[n] \cdot w[n]$$

- In the frequency domain

$$H_t(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\varphi})\Psi(e^{j(\omega-\varphi)})\,d\varphi$$

- where $H_t(e^{j\omega})$ and $\Psi(e^{j\omega})$ are the DTFTs of $h_t[n]$ and $w[n]$, respectively

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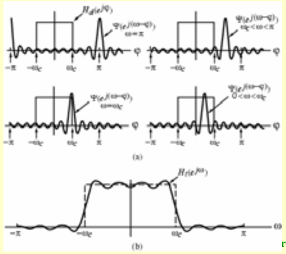
Gibbs Phenomenon

- Thus $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of $H_d(e^{j\omega})$ with $\Psi(e^{j\omega})$



Gibbs Phenomenon

- If $\Psi(e^{j\omega})$ is a very narrow pulse centered at $\omega = 0$ (ideally a delta function) compared to variations in $H_d(e^{j\omega})$, then $H_t(e^{j\omega})$ will approximate $H_d(e^{j\omega})$ very closely
- Length $2M+1$ of $w[n]$ should be very large
- On the other hand, length $2M+1$ of $h_t[n]$ should be as small as possible to reduce computational complexity

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Gibbs Phenomenon

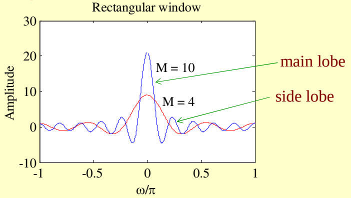
- A **rectangular window** is used to achieve simple truncation:

$$w_R[n] = \begin{cases} 1, & 0 \le |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

- Presence of oscillatory behavior in $H_t(e^{j\omega})$ is basically due to:
  - 1) $h_d[n]$ is infinitely long and not absolutely summable, and hence filter is unstable
  - 2) Rectangular window has an abrupt transition to zero

Gibbs Phenomenon

- Oscillatory behavior can be explained by examining the DTFT $\Psi_R(e^{j\omega})$ of $w_R[n]$:



- $\Psi_R(e^{j\omega})$ has a **main lobe** centered at $\omega = 0$
- Other ripples are called **sidelobes**

Gibbs Phenomenon

- Main lobe of $\Psi_R(e^{j\omega})$ characterized by its width $4\pi/(2M+1)$ defined by first zero crossings on both sides of $\omega = 0$
- As M increases, width of main lobe decreases as desired
- Area under each lobe remains constant while width of each lobe decreases with an increase in M
- Ripples in $H_t(e^{j\omega})$ around the point of discontinuity occur more closely but with no decrease in amplitude as M increases

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Gibbs Phenomenon

- Rectangular window has an abrupt transition to zero outside the range $-M \le n \le M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

  (1) Using a window that tapers smoothly to zero at each end, or

  (2) Providing a smooth transition from passband to stopband in the magnitude specifications