

## Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity

- Hann:**

$$w[n] = 0.5 + 0.5 \cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \leq n \leq M$$

- Hamming:**

$$w[n] = 0.54 + 0.46 \cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \leq n \leq M$$

- Blackman:**

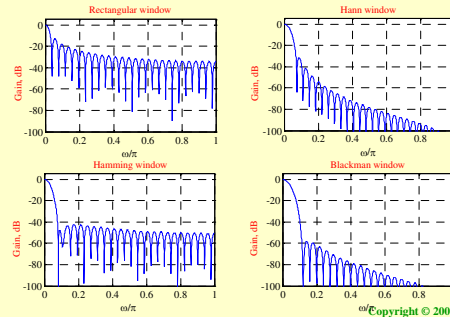
$$w[n] = 0.42 + 0.5 \cos\left(\frac{2\pi n}{2M+1}\right) + 0.08 \cos\left(\frac{4\pi n}{2M+1}\right)$$

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## Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for  $M = 25$  are shown below:



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## Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at  $\omega = 0$  followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:
  - Main lobe width**
  - Relative sidelobe level**

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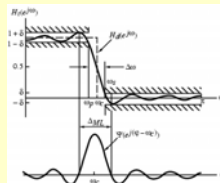
## Fixed Window Functions

- Main lobe width**  $\Delta_{ML}$  - given by the distance between zero crossings on both sides of main lobe
- Relative sidelobe level**  $A_{sl}$  - given by the difference in dB between amplitudes of largest sidelobe and main lobe

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## Fixed Window Functions



- Observe  $H_1(e^{j(\omega_c + \Delta\omega)}) + H_1(e^{j(\omega_c - \Delta\omega)}) \cong 1$
- Thus,  $H_1(e^{j\omega_c}) \cong 0.5$
- Passband and stopband ripples are the same

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## Fixed Window Functions

- Distance between the locations of the maximum passband deviation and minimum stopband value  $\cong \Delta_{ML}$
- Width of transition band
 
$$\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$$

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## Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main lobe width
- To reduce the passband and stopband ripple  $\delta$ , the area under the sidelobes should be very small
- Unfortunately, these two requirements are contradictory

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## Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency  $\omega_c$ , and is essentially constant

- In addition,

$$\Delta\omega \approx \frac{c}{M}$$

where  $c$  is a constant for most practical purposes

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## Fixed Window Functions

- Rectangular window** -  $\Delta_{ML} = 4\pi/(2M + 1)$   
 $A_{s\ell} = 13.3 \text{ dB}$ ,  $\alpha_s = 20.9 \text{ dB}$ ,  $\Delta\omega = 0.92\pi/M$
- Hann window** -  $\Delta_{ML} = 8\pi/(2M + 1)$   
 $A_{s\ell} = 31.5 \text{ dB}$ ,  $\alpha_s = 43.9 \text{ dB}$ ,  $\Delta\omega = 3.11\pi/M$
- Hamming window** -  $\Delta_{ML} = 8\pi/(2M + 1)$   
 $A_{s\ell} = 42.7 \text{ dB}$ ,  $\alpha_s = 54.5 \text{ dB}$ ,  $\Delta\omega = 3.32\pi/M$
- Blackman window** -  $\Delta_{ML} = 12\pi/(2M + 1)$   
 $A_{s\ell} = 58.1 \text{ dB}$ ,  $\alpha_s = 75.3 \text{ dB}$ ,  $\Delta\omega = 5.56\pi/M$

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## Fixed Window Functions

- Filter Design Steps -

- Set

$$\omega_c = (\omega_p + \omega_s)/2$$

- Choose window based on specified  $\alpha_s$

- Estimate  $M$  using

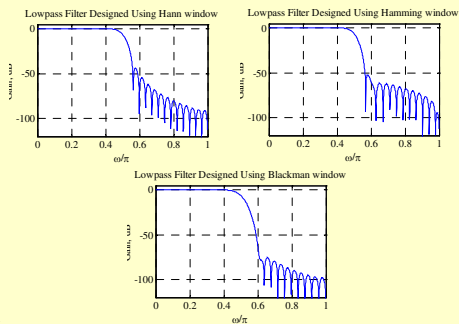
$$\Delta\omega \approx \frac{c}{M}$$

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## FIR Filter Design Example

- Lowpass filter of length 51 and  $\omega_c = \pi/2$



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## FIR Filter Design Example

- An increase in the main lobe width is associated with an increase in the width of the transition band
- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

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## Adjustable Window Functions

- **Dolph-Chebyshev Window** -

$$w[n] = \frac{1}{2M+1} \left[ \frac{1}{\gamma} + 2 \sum_{k=1}^M T_k \left( \beta \cos \frac{k}{2M+1} \right) \cos \frac{2nk\pi}{2M+1} \right], \quad -M \leq n \leq M$$

where  $\gamma = \frac{\text{amplitude of sidelobe}}{\text{main lobe amplitude}}$

$$\beta = \cosh \left( \frac{1}{2M} \cosh^{-1} \frac{1}{\gamma} \right)$$

and

$$T_\ell(x) = \begin{cases} \cos(\ell \cos^{-1} x), & |x| \leq 1 \\ \cosh(\ell \cosh^{-1} x), & |x| > 1 \end{cases}$$

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## Adjustable Window Functions

- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately

- Filter order is estimated using

$$N = \frac{2.056\alpha_s - 16.4}{2.85(\Delta\omega)}$$

where  $\Delta\omega$  is the normalized transition bandwidth, e.g, for a lowpass filter

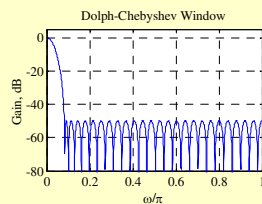
$$\Delta\omega = \omega_s - \omega_p$$

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## Adjustable Window Functions

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below



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## Adjustable Window Functions

Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height
- Stopband approximation error of filters designed have essentially equiripple behavior
- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

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## Adjustable Window Functions

- **Kaiser Window** -

$$w[n] = \frac{I_0 \{ \beta \sqrt{1 - (n/M)^2} \}}{I_0(\beta)}, \quad -M \leq n \leq M$$

where  $\beta$  is an adjustable parameter and  $I_0(u)$  is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty} \left[ \frac{(u/2)^r}{r!} \right]^2$$

- Note  $I_0(u) > 0$  for  $u > 0$
- In practice  $I_0(u) \cong 1 + \sum_{r=1}^{20} \left[ \frac{(u/2)^r}{r!} \right]^2$

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## Adjustable Window Functions

- $\beta$  controls the minimum stopband attenuation of the windowed filter response

- $\beta$  is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \leq \alpha_s \leq 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where  $\Delta\omega$  is the normalized transition bandwidth

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## FIR Filter Design Example

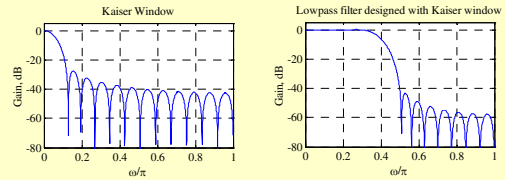
- **Specifications:**  $\omega_p = 0.3\pi$ ,  $\omega_s = 0.5\pi$ ,  $\alpha_s = 40$  dB
- **Thus**  $\omega_c = (\omega_p + \omega_s)/2 = 0.4\pi$   
 $\delta_s = 10^{-\alpha_s/20} = 0.01$   
 $\beta = 0.5842(19)^{0.4} + 0.07886 \times 19 = 3.3953$   
 $N = \frac{32}{2.285(0.2\pi)} = 22.2886$
- **Choose**  $N = 24$  implying  $M = 12$

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## FIR Filter Design Example

- **Hence**  $h_t[n] = \frac{\sin(0.4\pi n)}{\pi n} \cdot w[n]$ ,  $-12 \leq n \leq 12$   
**where**  $w[n]$  **is the**  $n$ -**th coefficient of a**  
**length-25 Kaiser window with**  $\beta = 3.3953$

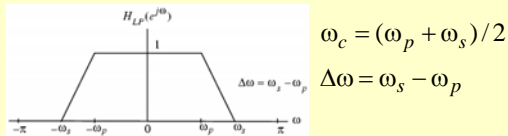


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## Impulse Responses of FIR Filters with a Smooth Transition

- **First-order spline passband-to-stopband transition**



$$h_{LP}[n] = \begin{cases} \omega_c / \pi, & n = 0 \\ \frac{2 \sin(\Delta\omega n / 2)}{\Delta\omega n} \cdot \frac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

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## Impulse Responses of FIR Filters with a Smooth Transition

- **Pth-order spline passband-to-stopband transition**

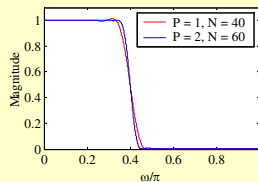
$$h_{LP}[n] = \begin{cases} \omega_c / \pi, & n = 0 \\ \left( \frac{2 \sin(\Delta\omega n / 2P)}{\Delta\omega n / 2P} \right)^P \cdot \frac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

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## Lowpass FIR Filter Design Example

- **Example**



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