Multirate Digital Signal Processing

Basic Sampling Rate Alteration Devices

- Up-sampler - Used to increase the sampling rate by an integer factor
- Down-sampler - Used to decrease the sampling rate by an integer factor

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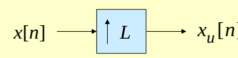
Up-Sampler

Time-Domain Characterization

- An up-sampler with an **up-sampling factor** L, where L is a positive integer, develops an output sequence $x_u[n]$ with a sampling rate that is L times larger than that of the input sequence $x[n]$
- Block-diagram representation



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Up-Sampler

- Up-sampling operation is implemented by inserting $L-1$ equidistant zero-valued samples between two consecutive samples of $x[n]$
- Input-output relation

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

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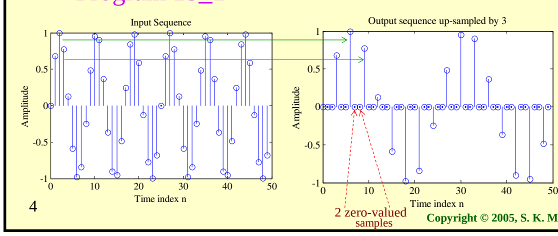
Up-Sampler

- Figure below shows the up-sampling by a factor of 3 of a sinusoidal sequence with a frequency of 0.12 Hz obtained using Program 13_1



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Up-Sampler

- In practice, the zero-valued samples inserted by the up-sampler are replaced with appropriate nonzero values using some type of filtering process
- Process is called **interpolation** and will be discussed later

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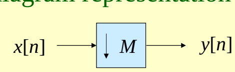
Down-Sampler

Time-Domain Characterization

- An down-sampler with a **down-sampling factor** M, where M is a positive integer, develops an output sequence $y[n]$ with a sampling rate that is $(1/M)$-th of that of the input sequence $x[n]$
- Block-diagram representation



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Down-Sampler

- Down-sampling operation is implemented by keeping every M-th sample of $x[n]$ and removing $M-1$ in-between samples to generate $y[n]$
- Input-output relation

$$y[n] = x[nM]$$

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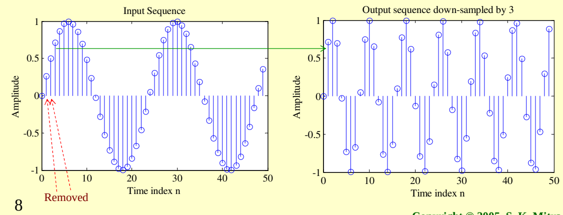
Down-Sampler

- Figure below shows the down-sampling by a factor of 3 of a sinusoidal sequence of frequency 0.042 Hz obtained using Program 13_2



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Basic Sampling Rate Alteration Devices

- Sampling periods have not been explicitly shown in the block-diagram representations of the up-sampler and the down-sampler
- This is for simplicity and the fact that the mathematical theory of multirate systems can be understood without bringing the sampling period T or the sampling frequency F_T into the picture

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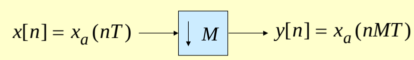
Down-Sampler

- Figure below shows explicitly the input-output sampling rates of the down-sampler



Input sampling frequency $F_T = \dfrac{1}{T}$

Output sampling frequency $F'_T = \dfrac{F_T}{M} = \dfrac{1}{T'}$

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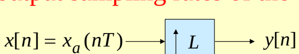
Up-Sampler

- Figure below shows explicitly the input-output sampling rates of the up-sampler



$$= \begin{cases} x_a(nT/L), & n=0,\pm L,\pm 2L,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Input sampling frequency $F_T = \dfrac{1}{T}$

Output sampling frequency $F'_T = LF_T = \dfrac{1}{T'}$

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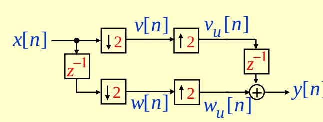
A Simple Multirate Structure



- The operation of the above multirate structure can be analyzed by writing down the relations between various signal variables, and the input as shown in the next slide

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A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$

A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$

A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$
$v_u[n]$:	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0	$x[8]$
$w_u[n]$:	$x[-1]$	0	$x[1]$	0	$x[3]$	0	$x[5]$	0	$x[7]$

A Simple Multirate Structure

n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$
$v_u[n]$:	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0	$x[8]$
$w_u[n]$:	$x[-1]$	0	$x[1]$	0	$x[3]$	0	$x[5]$	0	$x[7]$
$v_u[n-1]$:	0	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0
$y[n]$:	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$

$$y[n] = v_u[n-1] + w_u[n] = x[n-1]$$

Basic Sampling Rate Alteration Devices

- The up-sampler and the down-sampler are linear but time-varying discrete-time systems
- We illustrate the time-varying property of a down-sampler
- The time-varying property of an up-sampler can be proved in a similar manner

Basic Sampling Rate Alteration Devices

- Consider a factor-of-M down-sampler defined by $y[n] = x[nM]$
- Its output $y_1[n]$ for an input $x_1[n] = x[n-n_0]$ is then given by

$$y_1[n] = x_1[Mn] = x[Mn - n_0]$$

- From the input-output relation of the down-sampler we obtain

$$y[n-n_0] = x[M(n-n_0)]$$
$$= x[Mn - Mn_0] \neq y_1[n]$$

Up-Sampler

Frequency-Domain Characterization

- Consider first a factor-of-2 up-sampler whose input-output relation in the time-domain is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Up-Sampler

- In terms of the *z*-transform, the input-output relation is then given by

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-2m} = X(z^2)$$

Up-Sampler

- In a similar manner, we can show that for a factor-of-*L* up-sampler

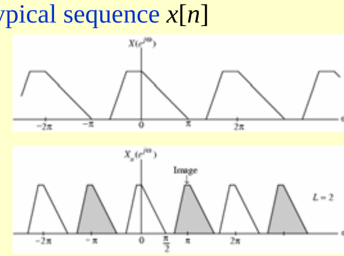
$$X_u(z) = X(z^L)$$

- On the unit circle, for $z = e^{j\omega}$, the input-output relation is given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

Up-Sampler

- Figure below shows the relation between $X(e^{j\omega})$ and $X_u(e^{j\omega})$ for $L = 2$ in the case of a typical sequence $x[n]$



Up-Sampler

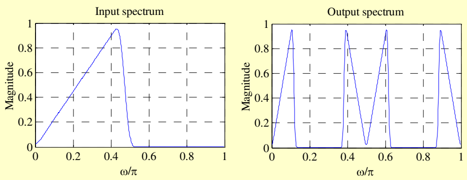
- As can be seen, a factor-of-2 sampling rate expansion leads to a compression of $X(e^{j\omega})$ by a factor of 2 and a 2-fold repetition in the baseband $[0, 2\pi]$
- This process is called imaging as we get an additional "**image**" of the input spectrum

Up-Sampler

- Similarly in the case of a factor-of-*L* sampling rate expansion, there will be $L-1$ additional images of the input spectrum in the baseband
- Lowpass filtering of $x_u[n]$ removes the $L-1$ images and in effect "fills in" the zero-valued samples in $x_u[n]$ with interpolated sample values

Up-Sampler

- Program 13_3 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $L = 4$



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Down-Sampler

Frequency-Domain Characterization

- Applying the z-transform to the input-output relation of a factor-of-M down-sampler

$$y[n] = x[Mn]$$

we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n}$$

- The expression on the right-hand side cannot be directly expressed in terms of $X(z)$

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Down-Sampler

- To get around this problem, define a new sequence $x_{\text{int}}[n]$:

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Then

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn] z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x_{\text{int}}[k] z^{-k/M} = X_{\text{int}}(z^{1/M})$$

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Down-Sampler

- Now, $x_{\text{int}}[n]$ can be formally related to $x[n]$ through

$$x_{\text{int}}[n] = c[n] \cdot x[n]$$

where

$$c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- A convenient representation of $c[n]$ is given by

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

where $W_M = e^{-j2\pi/M}$

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Down-Sampler

- Taking the z-transform of $x_{\text{int}}[n] = c[n] \cdot x[n]$ and making use of

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

we arrive at

$$X_{\text{int}}(z) = \sum_{n=-\infty}^{\infty} c[n] x[n] z^{-n} = \frac{1}{M} \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{M-1} W_M^{kn} \right) x[n] z^{-n}$$
$$= \frac{1}{M} \sum_{k=0}^{M-1} \left(\sum_{n=-\infty}^{\infty} x[n] W_M^{kn} z^{-n} \right) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z W_M^{-k}\right)$$

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Down-Sampler

- Hence,

$$Y(z) = X_{\text{int}}(z^{1/M})$$
$$= \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

- On the unit circle,

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

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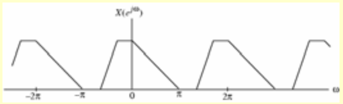
Down-Sampler

- Consider a factor-of-2 down-sampler with an input $x[n]$ whose spectrum is as shown below

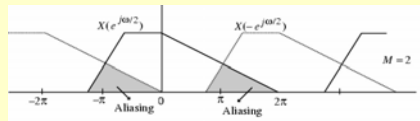


- The DTFTs of the output and the input sequences of this down-sampler are then related as

$$Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\}$$

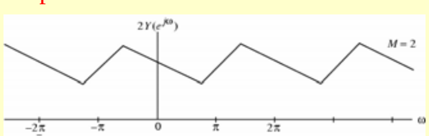
Down-Sampler

- Now $X(-e^{j\omega/2}) = X(e^{j(\omega-2\pi)/2})$ implying that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below



Down-Sampler

- The plots of the two terms have an overlap, and hence, in general, the original "shape" of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below



Down-Sampler

- This overlap causes the **aliasing** that takes place due to under-sampling
- There is no overlap, i.e., no aliasing, only if

$$X(e^{j\omega}) = 0 \quad \text{for } |\omega| \ge \pi/2$$

- Note: $Y(e^{j\omega})$ is indeed periodic with a period 2π, even though the stretched version of $X(e^{j\omega})$ is periodic with a period 4π

Down-Sampler

- For the general case, the relation between the DTFTs of the output and the input of a factor-of-M down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

- ➡ $Y(e^{j\omega})$ is a sum of M uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of $1/M$

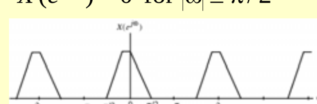
Down-Sampler

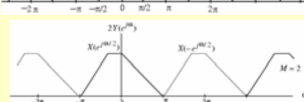
- Aliasing is absent if and only if

$$X(e^{j\omega}) = 0 \ \ \textit{for } |\omega| \ge \pi/M$$

as shown below for $M = 2$

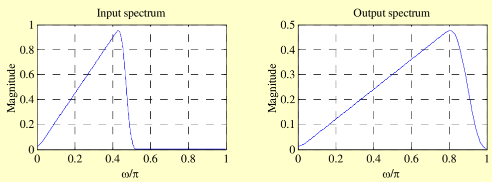
$$X(e^{j\omega}) = 0 \ \text{ for } |\omega| \ge \pi/2$$





Down-Sampler

- Program 13_4 can be used to illustrate the frequency-domain properties of the down-sampler shown below for $M = 2$



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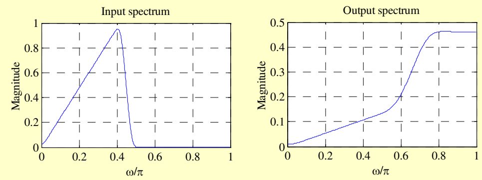
Down-Sampler

- The input and output spectra of a down-sampler with $M = 3$ obtained using Program 13_4 are shown below



- Effect of aliasing can be clearly seen

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Cascade Equivalences

- A complex multirate system is formed by an interconnection of the up-sampler, the down-sampler, and the components of an LTI digital filter
- In many applications these devices appear in a cascade form
- An interchange of the positions of the branches in a cascade often can lead to a computationally efficient realization

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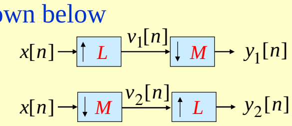
Up-Sampler and Down-sampler Cascade

- To implement a fractional change in the sampling rate we need to employ a cascade of an up-sampler and a down-sampler
- Consider the two cascade connections shown below



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Up-Sampler and Down-sampler Cascade

- Consider the top cascade shown in the previous slide
- Here, we have $V_1(z) = X(z^L)$

  and
  $$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(z^{1/M}W_M^{-k})$$
- Combining the last two equations we get
  $$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M}W_M^{-kL})$$

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Up-Sampler and Down-sampler Cascade

- We next consider the bottom cascade shown in Slide 36
- Here, we have
  $$V_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$$

  and $Y_2(z) = V_2(z^L)$
- Combining the last two equations we get
  $$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M}W_M^{-k})$$

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Up-Sampler and Down-sampler Cascade

- It follows from the above that $Y_1(z) = Y_2(z)$ if

$$\frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL}) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k})$$

- The above equality holds if and only if M and L are relatively prime, i.e. M and L do not have a common factor that is an integer $r > 1$, as then W_M^{-k} and W_M^{-kL} take the same set of values for $k = 0, 1, \ldots, M-1$

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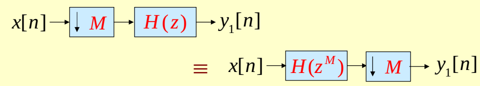
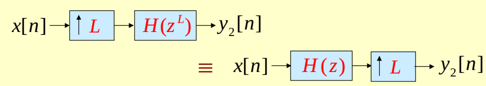
Noble Identities

- Two other cascade equivalences are shown below

Cascade equivalence #1



Cascade equivalence #2



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Multirate Structures for Sampling Rate Conversion

- From the sampling theorem it is known that the sampling rate of a critically sampled discrete-time signal with a spectrum occupying the full Nyquist range cannot be reduced any further since such a reduction will introduce aliasing
- Hence, the bandwidth of a critically sampled signal must be reduced by lowpass filtering before its sampling rate is reduced by a down-sampler

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Multirate Structures for Sampling Rate Conversion

- Likewise, the zero-valued samples introduced by an up-sampler must be interpolated to more appropriate values for an effective sampling rate increase
- We shall show shortly that this interpolation can be achieved simply by digital lowpass filtering

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Multirate Structures for Sampling Rate Conversion

- Since a fractional-rate sampling rate converter with a rational conversion factor can be realized by cascading an interpolator with a decimator, filters are also needed in the design of such multirate systems

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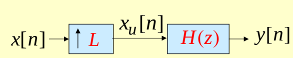
Basic Structures

- Since up-sampling by an integer factor L causes periodic repetition of the basic spectrum, the basic interpolator structure for integer-valued sampling rate increase consists of an up-sampler followed by a low-pass filter $H(z)$ with a cutoff at π/L as indicated below:



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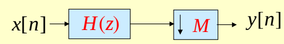
Basic Structures

- The lowpass filter $H(z)$, called the interpolation filter, removes the unwanted images in the spectra of the up-sampled signal $x_u[n]$
- On the other hand, down-sampling by an integer factor M may result in aliasing

Basic Structures

- Hence, the basic decimator structure for integer-valued sampling rate decrease consists of a lowpss filter $H(z)$ with a cutoff at π/M, followed by the down-sampler as shown below

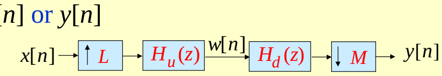


Basic Structures

- Here, the lowpass filter $H(z)$, called the decimation filter, bandlimits the input signal $x[n]$ to $|\omega| < \pi/M$ prior to down-sampling, to ensure no aliasing
- It can be shown that the transpose of a factor-of-M decimator is a factor-of-M interpolator

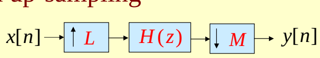
Basic Structures

- A fractional change in the sampling rate by a rational factor L/M can be achieved by cascading a factor-of-L interpolator with a factor-of-M decimator
- The interpolator must precede the decimator as shown below to ensure that the baseband of $w[n]$ is greater than or equal to that of $x[n]$ or $y[n]$



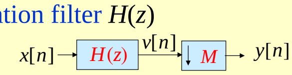
Basic Structures

- As both the interpolation filter $H_u(z)$ and the decimation filter $H_d(z)$ operate at the same sampling rate, they can be replaced with a single filter designed to avoid aliasing that may be caused by down-sampling and eliminate images resulting from up-sampling



Input-Output Relation of the Decimator

- For the decimator structure shown below, let $h[n]$ denote the impulse response of the decimation filter $H(z)$



- Then

$$v[n] = \sum_{\ell=-\infty}^{\infty} h[n-\ell]x[\ell]$$

and

$$y[n] = v[Mn]$$

Input-Output Relation of the Decimator

- Combining the last two equations we arrive at the desired input-output relation of the decimator given by

$$y[n] = \sum_{\ell=-\infty}^{\infty} h[Mn-\ell]x[\ell]$$

- In the z-domain, the input-output relation of the decimation filter is given by

$$V(z) = H(z)X(z)$$

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Input-Output Relation of the Decimator

- Now the input-output relation of the dowm-sampler is given by

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(z^{1/M}W_M^{-k})$$

- Combining the last two equations we arrive at the input-output relation of the decimator as

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(z^{1/M}W_M^{-k})X(z^{1/M}W_M^{-k})$$

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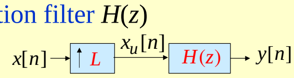
Input-Output Relation of the Interpolator

- For the interpolator structure shown below, let $h[n]$ denote the impulse response of the decimation filter $H(z)$



- Then

$$y[n] = \sum_{\ell=-\infty}^{\infty} h[n-\ell]x_u[\ell]$$

and

$$x_u[Lm] = x[m], \quad m = 0, \pm 1, \pm 2, \ldots$$

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Input-Output Relation of the Interpolator

- Combining the last two equations and making a change of a variable, we arrive at the desired time-domain input-output relation of the interpolator as

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-Lm]x[m]$$

- In the z-domain, the input-output relation of the interpolator is thus given by

$$Y(z) = H(z)X(z^L)$$

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Input-Output Relation of the Fractional-Rate Converter

- Here, in the time-domain the input-output relation is given by

$$y[n] = \sum_{m=-\infty}^{\infty} h[Mn-Lm]x[m]$$

- In the z-domain it is given by

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(z^{1/M}W_M^{-k})X(z^{L/M}W_M^{-kL})$$

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Interpolation Filter Specifications

- Assume $x[n]$ has been obtained by sampling a continuous-time signal $x_a(t)$ at the Nyquist rate
- If $X_a(j\Omega)$ and $X(e^{j\omega})$ denote the Fourier transforms of $x_a(t)$ and $x[n]$, respectively, then it can be shown

$$X(e^{j\omega}) = \frac{1}{T_o}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_o}\right)$$

- where T_o is the sampling period

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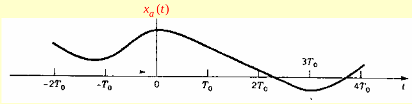
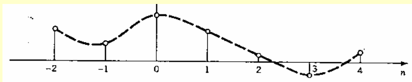
Interpolation Filter Specifications

- Figures below show $x_a(t)$ and $x[n]$ obtained by sampling $x_a(t)$ at the Nyquist rate





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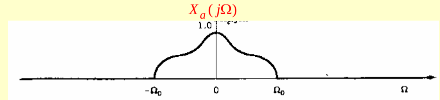
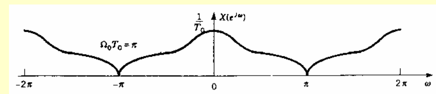
Interpolation Filter Specifications

- Figures below show the Fourier transforms of $x_a(t)$ and $x[n]$





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Interpolation Filter Specifications

- Since the sampling is being performed at the Nyquist rate, there is no overlap between the shifted spectras of $X(j\omega/T_o)$
- If we instead sample $x_a(t)$ at a much higher rate $T = T_o/L$ yielding $y[n]$, its Fourier transform $Y(e^{j\omega})$ is related to $X_a(j\Omega)$ through

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T}\right) = \frac{L}{T_o}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_o/L}\right)$$

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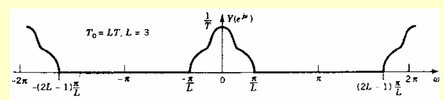
Interpolation Filter Specifications

- Figure below show the Fourier transform of $y[n]$



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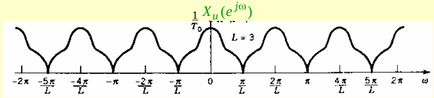
Interpolation Filter Specifications

- On the other hand, if we pass $x[n]$ through a factor-of-L up-sampler generating $x_u[n]$, the relation between the Fourier transforms of $x[n]$ and $x_u[n]$ are given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$



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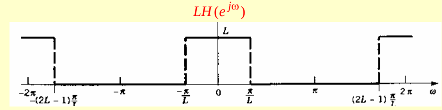
Interpolation Filter Specifications

- It therefore follows that if $x_u[n]$ is passed through an ideal lowpass filter $H(z)$ with a cutoff at π/L and a gain of L, the output of the filter will be precisely $y[n]$



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Interpolation Filter Specifications

- In practice, a transition band is provided to ensure the realizability and stability of the lowpass interpolation filter $H(z)$
- Hence, the desired lowpass filter should have a stopband edge at $\omega_s = \pi / L$ and a passband edge ω_p close to ω_s to reduce the distortion of the spectrum of $x[n]$

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Interpolation Filter Specifications

- If ω_c is the highest frequency that needs to be preserved in $x[n]$, then

$$\omega_p = \omega_c / L$$

- Summarizing the specifications of the lowpass interpolation filter are thus given by

$$\left|H(e^{j\omega})\right| = \begin{cases} L, & |\omega| \le \omega_c / L \\ 0, & \pi / L \le |\omega| \le \pi \end{cases}$$

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Decimation Filter Specifications

- In a similar manner, we can develop the specifications for the lowpass decimation filter that are given by

$$\left|H(e^{j\omega})\right| = \begin{cases} 1, & |\omega| \le \omega_c / M \\ 0, & \pi / M \le |\omega| \le \pi \end{cases}$$

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Filter Design Methods

- The design of the filter $H(z)$ is a standard IIR or FIR lowpass filter design problem
- Any one of the techniques outlined in Chapter 7 can be applied for the design of these lowpass filters

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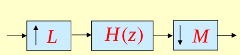
Filters for Fractional Sampling Rate Alteration

- For the fractional sampling rate structure shown below, the lowpass filter $H(z)$ has a stopband edge frequency given by

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$



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Computational Requirements

- The lowpass decimation or interpolation filter can be designed either as an FIR or an IIR digital filter
- In the case of single-rate digital signal processing, IIR digital filters are, in general, computationally more efficient than equivalent FIR digital filters, and are therefore preferred where computational cost needs to be minimized

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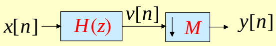
Computational Requirements

- This issue is not quite the same in the case of multirate digital signal processing
- To illustrate this point further, consider the factor-of-M decimator shown below



- If the decimation filter $H(z)$ is an FIR filter of length N implemented in a direct form, then

$$v[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

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Computational Requirements

- Now, the down-sampler keeps only every M-th sample of $v[n]$ at its output
- Hence, it is sufficient to compute $v[n]$ only for values of n that are multiples of M and skip the computations of in-between samples
- This leads to a factor of M savings in the computational complexity

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Computational Requirements

- Now assume $H(z)$ to be an IIR filter of order K with a transfer function

$$\frac{V(z)}{X(z)} = H(z) = \frac{P(z)}{D(z)}$$

where

$$P(z) = \sum_{n=0}^{K} p_n z^{-n}$$

$$D(z) = 1 + \sum_{n=1}^{K} d_n z^{-n}$$

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Computational Requirements

- Its direct form implementation is given by

$$w[n] = -d_1 w[n-1] - d_2 w[n-2] - \cdots - d_K w[n-K] + x[n]$$

$$v[n] = p_0 w[n] + p_1 w[n-1] + \cdots + p_K w[n-K]$$

- Since $v[n]$ is being down-sampled, it is sufficient to compute $v[n]$ only for values of n that are integer multiples of M

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Computational Requirements

- However, the intermediate signal $w[n]$ must be computed for all values of n
- For example, in the computation of

$$v[M] = p_0 w[M] + p_1 w[M-1] + \cdots + p_K w[M-K]$$

$K+1$ successive values of $w[n]$ are still required

- As a result, the savings in the computation in this case is going to be less than a factor of M

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Computational Requirements

- Example - We compare the computational complexity of various implementations of a factor-of-M decimator
- Let the sampling frequency be F_T
- Then the number of multiplications per second, to be denoted as $\mathcal{R}_M$, are as follows for various computational schemes

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Computational Requirements

- FIR $H(z)$ of length N:

$$\mathcal{R}_{M,FIR} = N \times F_T$$

- FIR $H(z)$ of length N followed by a down-sampler:

$$\mathcal{R}_{M,FIR-DEC} = N \times F_T / M$$

- IIR $H(z)$ of order K:

$$\mathcal{R}_{M,IIR} = (2K+1) \times F_T$$

- IIR $H(z)$ of order K followed by a down-sampler:

$$\mathcal{R}_{M,IIR-DEC} = K \times F_T + (K+1) \times F_T / M$$

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Computational Requirements

- In the FIR case, savings in computations is by a factor of M
- In the IIR case, savings in computations is by a factor of $M(2K+1)/[(M+1)K+1]$, which is not significant for large K
- For $M = 10$ and $K = 9$, the savings is only by a factor of 1.9
- There are certain cases where the IIR filter can be computationally more efficient

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Computational Requirements

- For the case of interpolator design, very similar arguments hold
- If $H(z)$ is an FIR interpolation filter, then the computational savings is by a factor of L (since $v[n]$ has $L-1$ zeros between its consecutive nonzero samples)
- On the other hand, computational savings is significantly less with IIR filters

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Sampling Rate Alteration Using MATLAB

- The function `decimate` can be employed to reduce the sampling rate of an input signal vector `x` by an integer factor `M` to generate the output signal vector `y`
- The decimation of a sequence by a factor of M can be obtained using Program 10_5 which employs the function `decimate`

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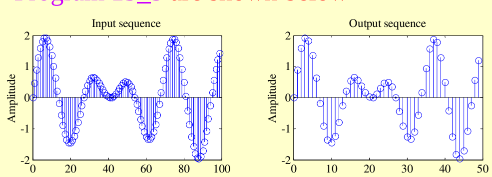
Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-2 decimator designed using the Program 13_5 are shown below



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Sampling Rate Alteration Using MATLAB

- The function `interp` can be employed to increase the sampling rate of an input signal `x` by an integer factor `L` generating the output vector `y`
- The lowpass filter designed by the M-file is a symmetric FIR filter

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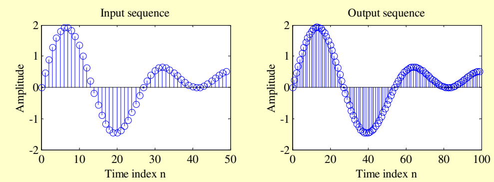
Sampling Rate Alteration Using MATLAB

- The filter allows the original input samples to appear as is in the output and finds the missing samples by minimizing the mean-square errors between these samples and their ideal values
- The interpolation of a sequence `x` by a factor of `L` can be obtained using the Program 13_6 which employs the function `interp`

Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-2 interpolator designed using Program 13_6 are shown below

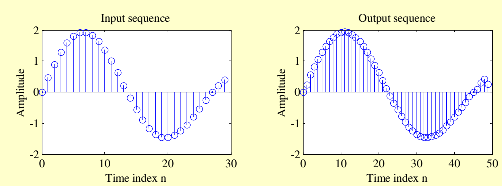


Sampling Rate Alteration Using MATLAB

- The function `resample` can be employed to increase the sampling rate of an input vector `x` by a ratio of two positive integers, `L/M`, generating an output vector `y`
- The M-file employs a lowpass FIR filter designed using `fir1` with a Kaiser window
- The fractional interpolation of a sequence can be obtained using Program 13_7 which employs the function `resample`

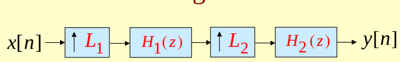
Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-5/3 interpolator designed using Program 13_7 are given below



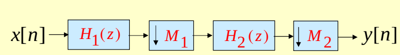
Multistage Design of Decimator and Interpolator

- The interpolator and the decimator can also be designed in more than one stages
- For example if the interpolation factor L can be expressed as a product of two integers, L_1 and L_2, then the factor-of-L interpolator can be realized in two stages as shown below



Multistage Design of Decimator and Interpolator

- Likewise if the decimator factor M can be expressed as a product of two integers, M_1 and M_2, then the factor-of-M interpolator can be realized in two stages as shown below

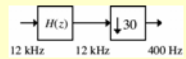


Multistage Design of Decimator and Interpolator

- Of course, the design can involve more than two stages, depending on the number of factors used to express L and M, respectively
- In general, the computational efficiency is improved significantly by designing the sampling rate alteration system as a cascade of several stages
- We consider the use of FIR filters here

Multistage Design of Decimator and Interpolator

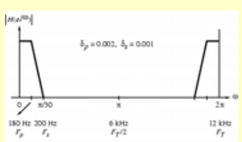
- Example - Consider the design of a decimator for reducing the sampling rate of a signal from 12 kHz to 400 Hz
- The desired down-sampling factor is therefore M = 30 as shown below



Multistage Design of Decimator and Interpolator

- Specifications for the decimation filter $H(z)$ are assumed to be as follows:

$$F_p = 180\,\text{Hz}, \quad F_s = 200\,\text{Hz},$$
$$\delta_p = 0.002, \quad \delta_s = 0.001$$



Multistage Design of Decimator and Interpolator

- Assume $H(z)$ to be designed as an equiripple linear-phase FIR filter
- Now Kaiser's formula for estimating the order of $H(z)$ to meet the specifications is given by

$$N = \frac{-20\log_{10}\sqrt{\delta_p\delta_s} - 13}{14.6\Delta f}$$

where $\Delta f = (F_s - F_p)/F_T$ is the normalized transition bandwidth

Multistage Design of Decimator and Interpolator

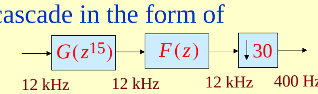
- The M-file kaiord determines the filter order using Kaiser's formula
- Using kaiord we obtain N = 1808
- Therefore, the number of multiplications per second in the single-stage implementation of the factor-of-30 decimator is

$$\mathcal{R}_{M,H} = 1809 \times \frac{12{,}000}{30} = 723{,}600$$

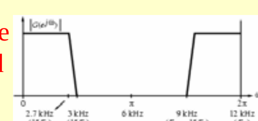
Multistage Design of Decimator and Interpolator

- We next implement $H(z)$ using the IFIR approach as a cascade in the form of

$G(z^{15})F(z)$

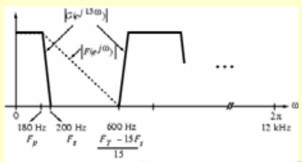


- The specifications of the parent filter $G(z)$ should thus be as shown on the right



Multistage Design of Decimator and Interpolator

- This corresponds to stretching the specifications of $H(z)$ by 15
- Figure below shows the magnitude response of $G(z^{15})$ and the desired response of $F(z)$



Multistage Design of Decimator and Interpolator

- Note: The desired response of $F(z)$ has a wider transition band as it takes into account the spectral gaps between the passbands of $G(z^{15})$
- Because of the cascade connection, the overall ripple of the cascade in dB is given by the sum of the passband ripples of $F(z)$ and $G(z^{15})$ in dB

Multistage Design of Decimator and Interpolator

- This can be compensated for by designing $F(z)$ and $G(z)$ to have a passband ripple of $\delta_p = 0.001$ each
- On the other hand, the cascade of $F(z)$ and $G(z^{15})$ has a stopband at least as good as $F(z)$ or $G(z^{15})$, individually
- So we can choose $\delta_s = 0.001$ for both filters

Multistage Design of Decimator and Interpolator

- Thus, specifications for the two filters $G(z)$ and $F(z)$ are as follows:

$$G(z):\ \delta_p = 0.001,\ \ \delta_s = 0.001,\ \ \Delta f = \frac{300}{12{,}000}$$

$$F(z):\ \delta_p = 0.001,\ \ \delta_s = 0.001,\ \ \Delta f = \frac{420}{12{,}000}$$

- The filter orders obtained using the M-file kaiord are: Order of $G(z)$ =129

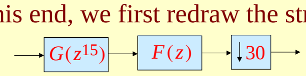
  Order of $F(z)$ = 92

Multistage Design of Decimator and Interpolator

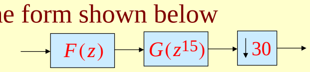
- The length of $H(z)$ for a direct implementation is 1809
- The length of cascade implementation $G(z^{15})F(z)$ is $92 + 15 \times 129 + 1 = 2028$
- ⇨ The length of the cascade structure is higher

Multistage Design of Decimator and Interpolator

- The computational complexity of the decimator implemented using the cascade structure can be dramatically reduced by making use of the cascade equivalence #1
- To this end, we first redraw the structure

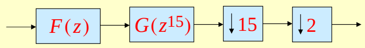


in the form shown below

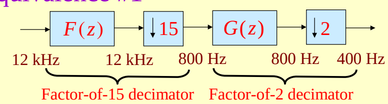


Multistage Design of Decimator and Interpolator

- The last structure is equivalent to the one shown below



- The above can be redrawn as indicated below by making use of the cascade equivalence #1



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Multistage Design of Decimator and Interpolator

- From the last realization we observe that the implementation of $G(z)$ followed by a factor-of-2 down-sampler requires

$$\mathcal{R}_{M,G} = 130 \times \frac{800}{2} = 52{,}000 \quad \text{mult/sec}$$

- Likewise, the implementation of $F(z)$ followed by a factor-of-15 down-sampler requires

$$\mathcal{R}_{M,F} = 93 \times \frac{12{,}000}{15} = 74{,}400 \quad \text{mult/sec}$$

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Multistage Design of Decimator and Interpolator

- The total complexity of the IFIR-based implementation of the factor-of-30 decimator is therefore

  52,000 + 74,400 = 126,400 mult/sec

  which is about 5.72 times smaller than that of a direct implementation of the decimation filter $H(z)$

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