Linear-Phase FIR Transfer Functions

- It is impossible to design an IIR transfer function with an exact linear-phase
- It is always possible to design an FIR transfer function with an exact linear-phase response
- We now develop the forms of the linear-phase FIR transfer function $H(z)$ with real impulse response $h[n]$

1

Linear-Phase FIR Transfer Functions

- Let $$H(z)=\sum_{n=0}^{N} h[n]z^{-n}$$
- If $H(z)$ is to have a linear-phase, its frequency response must be of the form
$$H(e^{j\omega})=e^{j(c\omega+\beta)}\breve{H}(\omega)$$
where c and β are constants, and $\breve{H}(\omega)$, called the amplitude response, also called the zero-phase response, is a real function of ω

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Linear-Phase FIR Transfer Functions

- For a real impulse response, the magnitude response $|H(e^{j\omega})|$ is an even function of ω, i.e.,
$$|H(e^{j\omega})|=|H(e^{-j\omega})|$$
- Since $|H(e^{j\omega})|=|\breve{H}(\omega)|$, the amplitude response is then either an even function or an odd function of ω, i.e.
$$\breve{H}(-\omega)=\pm\breve{H}(\omega)$$

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Linear-Phase FIR Transfer Functions

- The frequency response satisfies the relation
$$H(e^{j\omega})=H^{*}(e^{-j\omega})$$
or, equivalently, the relation
$$e^{j(c\omega+\beta)}\breve{H}(\omega)=e^{-j(-c\omega+\beta)}\breve{H}(-\omega)$$
- If $\breve{H}(\omega)$ is an even function, then the above relation leads to
$$e^{j\beta}=e^{-j\beta}$$
implying that either $\beta = 0$ or $\beta = \pi$

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Linear-Phase FIR Transfer Functions

- From
$$H(e^{j\omega})=e^{j(c\omega+\beta)}\breve{H}(\omega)$$
we have
$$\breve{H}(\omega)=e^{-j(c\omega+\beta)}H(e^{j\omega})$$
- Substituting the value of β in the above we get
$$\breve{H}(\omega)=\pm e^{-jc\omega}H(e^{j\omega})=\pm\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

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Linear-Phase FIR Transfer Functions

- Replacing ω with $-\omega$ in the previous equation we get
$$\breve{H}(-\omega)=\pm\sum_{\ell=0}^{N} h[\ell]e^{j\omega(c+\ell)}$$
- Making a change of variable $\ell=N-n$, we rewrite the above equation as
$$\breve{H}(-\omega)=\pm\sum_{n=0}^{N} h[N-n]e^{j\omega(c+N-n)}$$

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Linear-Phase FIR Transfer Functions

- As $\breve{H}(\omega) = \breve{H}(-\omega)$, we have

$$h[n]e^{-j\omega(c+n)} = h[N-n]e^{j\omega(c+N-n)}$$

- The above leads to the condition

$$h[n] = h[N-n], \qquad 0 \le n \le N$$

with $c = -N/2$

- Thus, the FIR filter with an even amplitude response will have a linear phase if it has a symmetric impulse response

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Linear-Phase FIR Transfer Functions

- If $\breve{H}(\omega)$ is an odd function of ω, then from

$$e^{j(c\omega+\beta)}\breve{H}(\omega) = e^{-j(-c\omega+\beta)}\breve{H}(-\omega)$$

we get $e^{j\beta} = -e^{-j\beta}$ as $\breve{H}(-\omega) = -\breve{H}(\omega)$

- The above is satisfied if $\beta = \pi/2$ or $\beta = -\pi/2$
- Then

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$$

reduces to

$$H(e^{j\omega}) = je^{jc\omega}\breve{H}(\omega)$$

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Linear-Phase FIR Transfer Functions

- The last equation can be rewritten as

$$\breve{H}(\omega) = -je^{-jc\omega}H(e^{j\omega}) = -j\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

- As $\breve{H}(-\omega) = -\breve{H}(\omega)$, from the above we get

$$\breve{H}(-\omega) = j\sum_{\ell=0}^{N} h[\ell]e^{j\omega(c+\ell)}$$

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Linear-Phase FIR Transfer Functions

- Making a change of variable $\ell = N - n$ we rewrite the last equation as

$$\breve{H}(-\omega) = j\sum_{\ell=0}^{N} h[\ell]e^{j\omega(c+\ell)}$$

- Equating the above with

$$\breve{H}(\omega) = -j\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

we arrive at the condition for linear phase as

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Linear-Phase FIR Transfer Functions

$$h[n] = -h[N-n], \qquad 0 \le n \le N$$

with $c = -N/2$

- Therefore, a FIR filter with an odd amplitude response will have linear-phase response if it has an antisymmetric impulse response

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Linear-Phase FIR Transfer Functions

- Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions
- For an antisymmetric FIR filter of odd length, i.e., N even

$$h[N/2] = 0$$

- We examine next the each of the 4 cases

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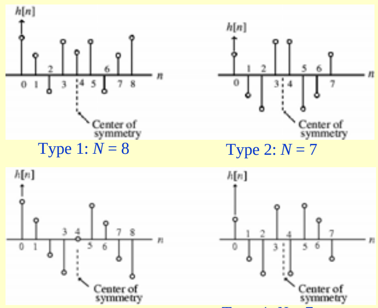
Linear-Phase FIR Transfer Functions



Type 1: $N = 8$ Type 2: $N = 7$

Type 3: $N = 8$ Type 4: $N = 7$

Linear-Phase FIR Transfer Functions

Type 1: Symmetric Impulse Response with Odd Length

- In this case, the degree N is even
- Assume $N = 8$ for simplicity
- The transfer function $H(z)$ is given by

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$
$$+ h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

Linear-Phase FIR Transfer Functions

- Because of symmetry, we have $h[0] = h[8]$, $h[1] = h[7]$, $h[2] = h[6]$, and $h[3] = h[5]$
- Thus, we can write

$$H(z) = h[0](1 + z^{-8}) + h[1](z^{-1} + z^{-7})$$
$$+ h[2](z^{-2} + z^{-6}) + h[3](z^{-3} + z^{-5}) + h[4]z^{-4}$$
$$= z^{-4}\{h[0](z^{4} + z^{-4}) + h[1](z^{3} + z^{-3})$$
$$+ h[2](z^{2} + z^{-2}) + h[3](z + z^{-1}) + h[4]\}$$

Linear-Phase FIR Transfer Functions

- The corresponding frequency response is then given by

$$H(e^{j\omega}) = e^{-j4\omega}\{2h[0]\cos(4\omega) + 2h[1]\cos(3\omega)$$
$$+ 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]\}$$

- The quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$

Linear-Phase FIR Transfer Functions

- The phase function here is given by

$$\theta(\omega) = -4\omega + \beta$$

  where β is either 0 or π, and hence, it is a linear function of ω
- The group delay is given by

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4$$

  indicating a constant group delay of 4 samples

Linear-Phase FIR Transfer Functions

- In the general case for Type 1 FIR filters, the frequency response is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2}\breve{H}(\omega)$$

  where the **amplitude response** $\breve{H}(\omega)$, also called the **zero-phase response**, is of the form

$$\breve{H}(\omega) = h[\tfrac{N}{2}] + 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\cos(\omega n)$$

Linear-Phase FIR Transfer Functions

- Example - Consider

$$H_0(z) = \frac{1}{6}[\tfrac{1}{2} + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \tfrac{1}{2}z^{-6}]$$

which is seen to be a slightly modified version of a length-7 moving-average FIR filter

- The above transfer function has a symmetric impulse response and therefore a linear phase response

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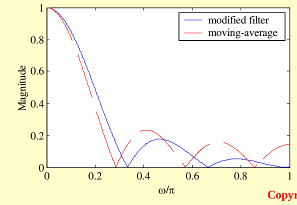
Linear-Phase FIR Transfer Functions

- A plot of the magnitude response of $H_0(z)$ along with that of the 7-point moving-average filter is shown below



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Linear-Phase FIR Transfer Functions

- Note the improved magnitude response obtained by simply changing the first and the last impulse response coefficients of a moving-average (MA) filter
- It can be shown that we an express

$$H_0(z) = \tfrac{1}{2}(1+z^{-1}) \cdot \tfrac{1}{6}(1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5})$$

which is seen to be a cascade of a 2-point MA filter with a 6-point MA filter

- Thus, $H_0(z)$ has a double zero at $z = -1$, i.e., $(\omega = \pi)$

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Linear-Phase FIR Transfer Functions

Type 2: Symmetric Impulse Response with Even Length

- In this case, the degree N is odd
- Assume $N = 7$ for simplicity
- The transfer function is of the form

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7}$$

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Linear-Phase FIR Transfer Functions

- Making use of the symmetry of the impulse response coefficients, the transfer function can be written as

$$H(z) = h[0](1+z^{-7}) + h[1](z^{-1}+z^{-6}) + h[2](z^{-2}+z^{-5}) + h[3](z^{-3}+z^{-4})$$

$$= z^{-7/2}\{h[0](z^{7/2}+z^{-7/2}) + h[1](z^{5/2}+z^{-5/2}) + h[2](z^{3/2}+z^{-3/2}) + h[3](z^{1/2}+z^{-1/2})\}$$

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Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j7\omega/2}\{2h[0]\cos(\tfrac{7\omega}{2}) + 2h[1]\cos(\tfrac{5\omega}{2}) + 2h[2]\cos(\tfrac{3\omega}{2}) + 2h[3]\cos(\tfrac{\omega}{2})\}$$

- As before, the quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$

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Linear-Phase FIR Transfer Functions

- Here the phase function is given by
$$\theta(\omega) = -\tfrac{7}{2}\omega + \beta$$
where again β is either 0 or π
- As a result, the phase is also a linear function of ω
- The corresponding group delay is
$$\tau(\omega) = \tfrac{7}{2}$$
indicating a group delay of $\tfrac{7}{2}$ samples

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Linear-Phase FIR Transfer Functions

- The expression for the frequency response in the general case for Type 2 FIR filters is of the form
$$H(e^{j\omega}) = e^{-jN\omega/2}\breve{H}(\omega)$$
where the amplitude response is given by
$$\breve{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n]\cos(\omega(n - \tfrac{1}{2}))$$

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Linear-Phase FIR Transfer Functions

Type 3: Antiymmetric Impulse Response with Odd Length

- In this case, the degree N is even
- Assume $N = 8$ for simplicity
- Applying the symmetry condition we get
$$H(z) = z^{-4}\{h[0](z^4 - z^{-4}) + h[1](z^3 - z^{-3}) + h[2](z^2 - z^{-2}) + h[3](z - z^{-1})\}$$

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Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by
$$H(e^{j\omega}) = e^{-j4\omega}e^{j\pi/2}\{2h[0]\sin(4\omega) + 2h[1]\sin(3\omega) + 2h[2]\sin(2\omega) + 2h[3]\sin(\omega)\}$$
- It also exhibits a linear phase response given by
$$\theta(\omega) = -4\omega + \tfrac{\pi}{2} + \beta$$
where β is either 0 or π

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Linear-Phase FIR Transfer Functions

- The group delay here is
$$\tau(\omega) = 4$$
indicating a constant group delay of 4 samples
- In the general case
$$H(e^{j\omega}) = je^{-jN\omega/2}\breve{H}(\omega)$$
where the amplitude response is of the form
$$\breve{H}(\omega) = 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\sin(\omega n)$$

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Linear-Phase FIR Transfer Functions

Type 4: Antiymmetric Impulse Response with Even Length

- In this case, the degree N is even
- Assume $N = 7$ for simplicity
- Applying the symmetry condition we get
$$H(z) = z^{-7/2}\{h[0](z^{7/2} - z^{-7/2}) + h[1](z^{5/2} - z^{-5/2}) + h[2](z^{3/2} - z^{-3/2}) + h[3](z^{1/2} - z^{-1/2})\}$$

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Linear-Phase FIR Transfer Functions

- The corresponding frequency response is given by

$$H(e^{j\omega}) = e^{-j7\omega/2} e^{j\pi/2}\{2h[0]\sin(\tfrac{7\omega}{2}) + 2h[1]\sin(\tfrac{5\omega}{2}) + 2h[2]\sin(\tfrac{3\omega}{2}) + 2h[3]\sin(\tfrac{\omega}{2})\}$$

- It again exhibits a linear phase response given by

$$\theta(\omega) = -\tfrac{7}{2}\omega + \tfrac{\pi}{2} + \beta$$

where β is either 0 or π

Linear-Phase FIR Transfer Functions

- The group delay is constant and is given by

$$\tau(\omega) = \tfrac{7}{2}$$

- In the general case we have

$$H(e^{j\omega}) = je^{-jN\omega/2}\breve{H}(\omega)$$

where now the amplitude response is of the form

$$\breve{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2} - n]\sin(\omega(n - \tfrac{1}{2}))$$

Linear-Phase FIR Transfer Functions

General Form of Frequency Response

- In each of the four types of linear-phase FIR filters, the frequency response is of the form

$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\beta}\breve{H}(\omega)$$

- The amplitude response $\breve{H}(\omega)$ for each of the four types of linear-phase FIR filters can become negative over certain frequency ranges, typically in the stopband

Linear-Phase FIR Transfer Functions

- **Example** – Consider the causal Type 1 FIR transfer function

$$H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6}$$

- Its amplitude and phase responses are given by

$$\breve{H}_1(\omega) = 6 - 6\cos(\omega) + 4\cos(2\omega) - 2\cos(3\omega)$$

$$\theta_1(\omega) = -3\omega$$

Linear-Phase FIR Transfer Functions

- Next, consider the causal Type 1 FIR transfer function

$$H_2(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6}$$

- Its amplitude and phase responses are given by

$$\breve{H}_2(\omega) = -\breve{H}_1(\omega)$$

$$\theta_2(\omega) = -3\omega + \pi$$

Linear-Phase FIR Transfer Functions

- Next, consider the causal Type 1 FIR transfer function

$$H_2(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6}$$

- Its amplitude and phase responses are given by

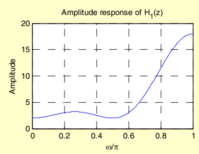
$$\breve{H}_2(\omega) = -\breve{H}_1(\omega)$$

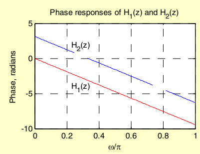
$$\theta_2(\omega) = -3\omega + \pi$$

- Note: $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$

Linear-Phase FIR Transfer Functions

- Hence, $H_1(z)$ and $H_2(z)$ have identical magnitude responses but phase responses differing by π as shown below





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Linear-Phase FIR Transfer Functions

- **Example** – Consider the causal Type 1 FIR transfer function

$$H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6}$$

- Its amplitude and phase responses are given by

$$\breve{H}_3(\omega) = -6\sin(\omega) + 4\sin(2\omega) + 2\sin(3\omega)$$

$$\theta_3(\omega) = -3\omega + \frac{\pi}{2}$$

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Linear-Phase FIR Transfer Functions

- Next, consider the causal Type 1 FIR transfer function

$$H_4(z) = -1 + 2z^{-1} - 3z^{-2} + 3z^{-4} - 2z^{-5} + z^{-6}$$

- Its amplitude and phase responses are given by

$$\breve{H}_4(\omega) = -\breve{H}_3(\omega)$$

$$\theta_4(\omega) = -3\omega - \frac{\pi}{2}$$

- Note: $|H_3(e^{j\omega})| = |H_4(e^{j\omega})|$

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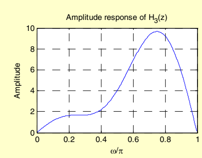
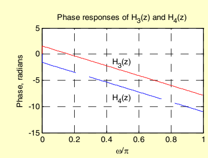
Linear-Phase FIR Transfer Functions

- Hence, $H_3(z)$ and $H_4(z)$ have identical magnitude responses but phase responses differing by π as shown below





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Linear-Phase FIR Transfer Functions

- The magnitude and phase responses of the linear-phase FIR are given by

$$|H(e^{j\omega})| = |\breve{H}(\omega)|$$

$$\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta, & \text{for } \breve{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi, & \text{for } \breve{H}(\omega) < 0 \end{cases}$$

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Linear-Phase FIR Transfer Functions

- The group delay in each case is

$$\tau(\omega) = \frac{N}{2}$$

- Note that, even though the group delay is constant, since in general $|H(e^{j\omega})|$ is not a constant, the output waveform is not a replica of the input waveform

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Linear-Phase FIR Transfer Functions

- Note that, even though the group delay is constant, since in general is not a constant, the output waveform is not a replica of the input waveform
- An FIR filter with a frequency response that is a real function of ω is often called a **zero-phase filter**
- Such a filter must have a noncausal impulse response

Zero Locations of Linear-Phase FIR Transfer Functions

- Consider first an FIR filter with a symmetric impulse response: $h[n] = h[N-n]$
- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = \sum_{n=0}^{N} h[N-n]z^{-n}$$

- By making a change of variable $m = N - n$, we can write

$$\sum_{n=0}^{N} h[N-n]z^{-n} = \sum_{m=0}^{N} h[m]z^{-N+m} = z^{-N}\sum_{m=0}^{N} h[m]z^{m}$$

Zero Locations of Linear-Phase FIR Transfer Functions

- But,

$$\sum_{m=0}^{N} h[m]z^{m} = H(z^{-1})$$

- Hence for an FIR filter with a symmetric impulse response of length N+1 we have

$$H(z) = z^{-N}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **mirror-image polynomial** (MIP)

Zero Locations of Linear-Phase FIR Transfer Functions

- Now consider first an FIR filter with an antisymmetric impulse response:

$$h[n] = -h[N-n]$$

- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = -\sum_{n=0}^{N} h[N-n]z^{-n}$$

- By making a change of variable $m = N - n$, we get

$$-\sum_{n=0}^{N} h[N-n]z^{-n} = -\sum_{m=0}^{N} h[m]z^{-N+m} = -z^{-N}H(z^{-1})$$

Zero Locations of Linear-Phase FIR Transfer Functions

- Hence, the transfer function $H(z)$ of an FIR filter with an antisymmetric impulse response satisfies the condition

$$H(z) = -z^{-N}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **antimirror-image polynomial** (AIP)

Zero Locations of Linear-Phase FIR Transfer Functions

- It follows from the relation $H(z) = \pm z^{-N}H(z^{-1})$ that if $z = \xi_o$ is a zero of $H(z)$, so is $z = 1/\xi_o$
- Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs
- Hence, a zero at $z = \xi_o$ is associated with a zero at $z = \xi_o^*$

Zero Locations of Linear-Phase FIR Transfer Functions

- Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by

$$z = re^{\pm j\phi}, \quad z = \tfrac{1}{r}e^{\pm j\phi}$$

- A zero on the unit circle appear as a pair

$$z = e^{\pm j\phi}$$

as its reciprocal is also its complex conjugate

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Zero Locations of Linear-Phase FIR Transfer Functions

- Since a zero at $z = \pm 1$ is its own reciprocal, it can appear only singly
- Now a Type 2 FIR filter satisfies

$$H(z) = z^{-N} H(z^{-1})$$

with degree N odd

- Hence $H(-1) = (-1)^{-N} H(-1) = -H(-1)$ implying $H(-1) = 0$, i.e., $H(z)$ must have a zero at $z = -1$

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Zero Locations of Linear-Phase FIR Transfer Functions

- Likewise, a Type 3 or 4 FIR filter satisfies

$$H(z) = -z^{-N} H(z^{-1})$$

- Thus $H(1) = -(1)^{-N} H(1) = -H(1)$ implying that $H(z)$ must have a zero at $z = 1$
- On the other hand, only the Type 3 FIR filter is restricted to have a zero at $z = -1$ since here the degree N is even and hence,

$$H(-1) = -(-1)^{-N} H(-1) = -H(-1)$$

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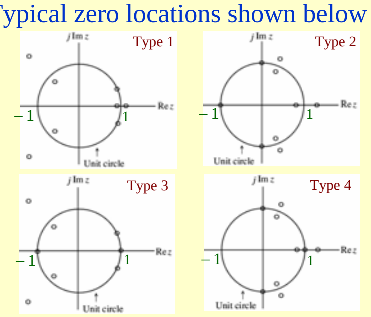
Zero Locations of Linear-Phase FIR Transfer Functions

- Typical zero locations shown below



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Zero Locations of Linear-Phase FIR Transfer Functions

- Summarizing

(1) Type 1 FIR filter: Either an even number or no zeros at $z = 1$ and $z = -1$

(2) Type 2 FIR filter: Either an even number or no zeros at $z = 1$, and an odd number of zeros at $z = -1$

(3) Type 3 FIR filter: An odd number of zeros at $z = 1$ and $z = -1$

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Zero Locations of Linear-Phase FIR Transfer Functions

(4) Type 4 FIR filter: An odd number of zeros at $z = 1$, and either an even number or no zeros at $z = -1$

- The presence of zeros at $z = \pm 1$ leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters

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Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero $z = -1$
- A Type 3 FIR filter has zeros at both $z = 1$ and $z = -1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter

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Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 4 FIR filter is not appropriate to design lowpass and bandstop filters due to the presence of a zero at $z = 1$
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter

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