Spectral Transformations of IIR Digital Filters

- Objective - Transform a given lowpass digital transfer function $G_L(z)$ to another digital transfer function $G_D(\hat{z})$ that could be a lowpass, highpass, bandpass or bandstop filter
- z^{-1} has been used to denote the unit delay in the prototype lowpass filter $G_L(z)$ and $\hat{z}^{-1}$ to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion

1

Spectral Transformations of IIR Digital Filters

- Unit circles in z- and $\hat{z}$-planes defined by

$$z = e^{j\omega}, \quad \hat{z} = e^{j\hat{\omega}}$$

- Transformation from z-domain to $\hat{z}$-domain given by

$$z = F(\hat{z})$$

- Then

$$G_D(\hat{z}) = G_L\{F(\hat{z})\}$$

2

Spectral Transformations of IIR Digital Filters

- From $z = F(\hat{z})$, thus $|z| = |F(\hat{z})|$, hence

$$|F(\hat{z})| \begin{cases} >1, & \text{if } |z|>1 \\ =1, & \text{if } |z|=1 \\ <1, & \text{if } |z|<1 \end{cases}$$

- Recall that a stable allpass function $A(z)$ satisfies the condition

3

Spectral Transformations of IIR Digital Filters

$$|A(z)| \begin{cases} <1, & \text{if } |z|>1 \\ =1, & \text{if } |z|=1 \\ >1, & \text{if } |z|<1 \end{cases}$$

- Therefore $1/F(\hat{z})$ must be a stable allpass function whose general form is

$$\frac{1}{F(\hat{z})} = \pm \prod_{\ell=1}^{L} \left(\frac{1-\alpha_\ell^* \hat{z}}{\hat{z}-\alpha_\ell} \right), \quad |\alpha_\ell| < 1$$

4

Lowpass-to-Lowpass Spectral Transformation

- To transform a lowpass filter $G_L(z)$ with a cutoff frequency ω_c to another lowpass filter $G_D(\hat{z})$ with a cutoff frequency $\hat{\omega}_c$, the transformation is

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1-\alpha\hat{z}}{\hat{z}-\alpha}$$

where α is a function of the two specified cutoff frequencies

5

Lowpass-to-Lowpass Spectral Transformation

- On the unit circle we have

$$e^{-j\omega} = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}}$$

- From the above we get

$$e^{-j\omega} \mp 1 = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}} \mp 1 = (1 \pm \alpha) \cdot \frac{e^{-j\hat{\omega}} \mp 1}{1 - \alpha e^{-j\hat{\omega}}}$$

- Taking the ratios of the above two expressions

$$\tan(\omega/2) = \left(\frac{1+\alpha}{1-\alpha} \right) \tan(\hat{\omega}/2)$$

6

Lowpass-to-Lowpass Spectral Transformation

- Solving we get $\alpha = \dfrac{\sin((\omega_c - \hat{\omega}_c)/2)}{\sin((\omega_c + \hat{\omega}_c)/2)}$
- Example - Consider the lowpass digital filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

which has a passband from dc to 0.25π with a 0.5 dB ripple

- Redesign the above filter to move the passband edge to 0.35π

7

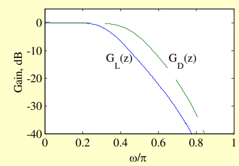
Lowpass-to-Lowpass Spectral Transformation

- Here

$$\alpha = -\frac{\sin(0.05\pi)}{\sin(0.3\pi)} = -0.1934$$

- Hence, the desired lowpass transfer function is

$$G_D(\hat{z}) = G_L(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1}+0.1934}{1+0.1934\hat{z}^{-1}}}$$



8

Lowpass-to-Lowpass Spectral Transformation

- The lowpass-to-lowpass transformation

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1-\alpha\hat{z}}{\hat{z}-\alpha}$$

can also be used as highpass-to-highpass, bandpass-to-bandpass and bandstop-to-bandstop transformations

9

Lowpass-to-Highpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}$$

- The transformation parameter α is given by

$$\alpha = -\frac{\cos((\omega_c + \hat{\omega}_c)/2)}{\cos((\omega_c - \hat{\omega}_c)/2)}$$

where ω_c is the cutoff frequency of the lowpass filter and $\hat{\omega}_c$ is the cutoff frequency of the desired highpass filter

10

Lowpass-to-Highpass Spectral Transformation

- Example - Transform the lowpass filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

- with a passband edge at 0.25π to a highpass filter with a passband edge at 0.55π
- Here $\alpha = -\cos(0.4\pi)/\cos(0.15\pi) = -0.3468$
- The desired transformation is

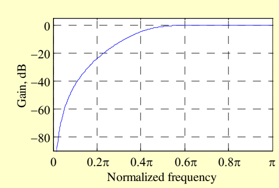
$$z^{-1} = -\frac{\hat{z}^{-1}-0.3468}{1-0.3468\hat{z}^{-1}}$$

11

Lowpass-to-Highpass Spectral Transformation

- The desired highpass filter is

$$G_D(\hat{z}) = G(z)\Big|_{z^{-1} = -\frac{\hat{z}^{-1}-0.3468}{1-0.3468\hat{z}^{-1}}}$$



12

Lowpass-to-Highpass Spectral Transformation

- The lowpass-to-highpass transformation can also be used to transform a highpass filter with a cutoff at ω_c to a lowpass filter with a cutoff at $\hat{\omega}_c$

  and transform a bandpass filter with a center frequency at ω_o to a bandstop filter with a center frequency at $\hat{\omega}_o$

13

Lowpass-to-Bandpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + \frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + 1}$$

14

Lowpass-to-Bandpass Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2)}{\cos((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2)}$$

$$\beta = \cot((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandpass filter

15

Lowpass-to-Bandpass Spectral Transformation

- Special Case - The transformation can be simplified if $\omega_c = \hat{\omega}_{c2} - \hat{\omega}_{c1}$
- Then the transformation reduces to

$$z^{-1} = -\hat{z}^{-1}\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}$$

where $\alpha = \cos\hat{\omega}_o$ with $\hat{\omega}_o$ denoting the desired center frequency of the bandpass filter

16

Lowpass-to-Bandstop Spectral Transformation

- Desired transformation

$$z^{-1} = \frac{\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + \frac{1-\beta}{1+\beta}}{\frac{1-\beta}{1+\beta}\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + 1}$$

17

Lowpass-to-Bandstop Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2)}{\cos((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2)}$$

$$\beta = \tan((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandstop filter

18

Generation of Allpass Function Using MATLAB

- The allpass function needed for the spectral transformation from a specified lowpass transfer function to a desired highpass or bandpass or bandstop transfer function can be generated using MATLAB

19

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Highpass Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2hp(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the highpass filter

20

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Bandpass Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2bp(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the bandpass filter

21

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Bandstop Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2bs(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the bandstop filter

22

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Highpass Example –**
  wold = 0.25π, wnew = 0.55π
- The MATLAB statement

```
[APnum, APden]
= allpasslp2hp(0.25, 0.55)
```

yields the mapping

$$z^{-1} \to \frac{-z^{-1} + 0.3468}{-0.3468z^{-1} + 1}$$

23

Spectral Transformation Using MATLAB

- The pertinent M-files are `iirlp2lp`, `iirlp2hp`, `iirlp2bp`, and `iirlp2bs`
- **Lowpass-to-Highpass Example –**

$$G_{LP}(z) = \frac{0.066(1+z^{-1})^3}{1-0.9353z^{-1}+0.5669z^{-2}-0.1015z^{-3}}$$

Passband edge wold = 0.25π

Desired passband edge of highpass filter wnew = 0.55π

24

Spectral Transformation Using MATLAB

- The MATLAB code fragments used are

```
b = 0.066*[1 3 3 1];
a = [1.00  -0.9353  0.5669 -0.1015];
[num,den,APnum,APden]
= iirlp2hp(b,a,0.25,0.55);
```

- The desired highpass filter obtained is

$$G_{HP}(z) = \frac{0.218(1-z^{-1})^3}{1-0.3521z^{-1}+0.3661z^{-2}-0.0329z^{-3}}$$

25

IIR Digital Filter Design Using MATLAB

- Order Estimation -
- For IIR filter design using bilinear transformation, MATLAB statements to determine the order and bandedge are:

  [N, Wn] = buttord(Wp, Ws, Rp, Rs);

  [N, Wn] = cheb1ord(Wp, Ws, Rp, Rs);

  [N, Wn] = cheb2ord(Wp, Ws, Rp, Rs);

  [N, Wn] = ellipord(Wp, Ws, Rp, Rs);

26

IIR Digital Filter Design Using MATLAB

- Example - Determine the minimum order of a Type 2 Chebyshev digital highpass filter with the following specifications:

  $F_p = 1$ kHz, $F_s = 0.6$ kHz, $F_T = 4$ kHz,

  $\alpha_p = 1$ dB, $\alpha_s = 40$ dB
- Here, $\text{Wp} = 2\times 1/4 = 0.5$, $\text{Ws} = 2\times 0.6/4 = 0.3$
- Using the statement

  [N, Wn] = cheb2ord(0.5, 0.3, 1, 40);

  we get N = 5 and Wn = 0.3224

27

IIR Digital Filter Design Using MATLAB

- Filter Design -
- For IIR filter design using bilinear transformation, MATLAB statements to use are:

  [b, a] = butter(N, Wn)

  [b, a] = cheby1(N, Rp, Wn)

  [b, a] = cheby2(N, Rs, Wn)

  [b, a] = ellip(N, Rp, Rs, Wn)

28

IIR Digital Filter Design Using MATLAB

- The form of transfer function obtained is

$$G(z) = \frac{B(z)}{A(z)} = \frac{b(1)+b(2)z^{-1}+\cdots+b(N+1)z^{-N}}{1+a(2)z^{-1}+\cdots+a(N+1)z^{-N}}$$

- The frequency response can be computed using the M-file freqz(b, a, w) where w is a set of specified angular frequencies
- It generates a set of complex frequency response samples from which magnitude and/or phase response samples cn be computed

29

IIR Digital Filter Design Using MATLAB

- Example - Design an elliptic IIR lowpass filter with the specifications: $F_p = 0.8$ kHz, $F_s = 1$ kHz, $F_T = 4$ kHz, $\alpha_p = 0.5$dB, $\alpha_s = 40$ dB
- Here, $\omega_p = 2\pi F_p/F_T = 0.4\pi$, $\omega_s = 2\pi F_s/F_T = 0.5\pi$
- Code fragments used are:

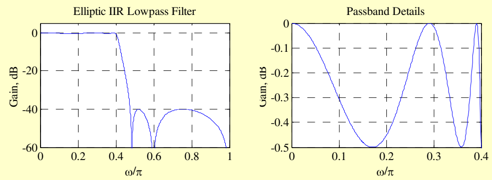
  [N,Wn] = ellipord(0.4, 0.5, 0.5, 40);

  [b, a] = ellip(N, 0.5, 40, Wn);

30

IIR Digital Filter Design Using MATLAB

- Gain response plot is shown below:



31

Computer-Aided Design of IIR Digital Filter

- The IIR filter design algorithms discussed so far are used in applications requiring filters with a frequency-selective magnitude response having either a lowpass or a highpass or a bandpass or a bandstop characteristics
- Designing IIR filters with other types of frequency responses usually involve the use of some type of iterative optimization techniques that are used to minimize the error between the desired response and that of the computer-generated filter

32

Computer-Aided Design of IIR Digital Filter

- **Basic Idea -**
- Let $H(e^{j\omega})$ denote the frequency response of the computer generated transfer function $H(z)$
- Let $D(e^{j\omega})$ denote the desired frequency response
- Objective is to design $H(z)$ so that $H(e^{j\omega})$ approximates $D(e^{j\omega})$ in some sense

33

Computer-Aided Design of IIR Digital Filter

- Usually the difference between $H(e^{j\omega})$ and $D(e^{j\omega})$ specified as a weighted error function $\mathcal{E}(\omega)$

$$\mathcal{E}(\omega) = W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]$$

is minimized for all values of ω over closed subintervals of $0 \le \omega \le \pi$

where $W(e^{j\omega})$ is some user-specified positive weighting function

34

Computer-Aided Design of IIR Digital Filter

- A commonly used approximation measure, called Chebyshev or minimax criterion, is to minimize the peak absolute value of $\mathcal{E}(\omega)$ given by

$$\varepsilon = \max_{\omega \in R} |\mathcal{E}(\omega)|$$

where R is the set of disjoint frequency bands in the range $0 \le \omega \le \pi$, on which the desired frequency response is defined

35

Computer-Aided Design of IIR Digital Filter

- In filtering applications, R is composed of the desired passbands and stopbands of the filter to be designed
- For example, for a lowpass filter design, R is the disjoint union of the frequency ranges $[0, \omega_p]$ and $[0, \omega_s]$, where ω_p and ω_s are, respectively, the passband and stopband edges

36

Computer-Aided Design of IIR Digital Filter

- Another approximation measure, called the least-p criterion, is to minimize the integral of the p-th power of $\mathcal{E}(\omega)$:

$$\varepsilon = \int_{\omega \in R} \left| W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})] \right|^p d\omega$$

over the specified frequency range R with p a positive integer

37

Computer-Aided Design of IIR Digital Filter

- The least-squares criterion obtained with $p = 2$ is often used for simplicity
- It can be shown that as $p \to \infty$, the least p-th solution approaches the minimax solution
- In practice, the integral error measure is approximated by a finite sum given by

$$\varepsilon = \sum_{i=1}^{K} \left| W(e^{j\omega_i})[H(e^{j\omega_i}) - D(e^{j\omega_i})] \right|^p$$

where ω_i, $1 \le i \le K$, is a suitably chosen dense grid of digital angular frequencies

38

Group Delay Equalization of IIR Digital Filter

- For a distortion-free transmission of an input signal in a prescribed frequency range through a digital filter, the transfer function of the filter should exhibit a unity magnitude response and a linear-phase response, i.e., a constant group delay, in the frequency band of interest

39

Group Delay Equalization of IIR Digital Filter

- The IIR digital filter design methods discussed so far lead to transfer functions with nonlinear phase responses
- Thus, to arrive at a frequency selective IIR digital filter with a constant group delay, a practical approach is to cascade the IIR digital filter meeting the magnitude response specifications with an allpass filter so that the overall group delay is a constant group delay in the band of interest

40

Group Delay Equalization of IIR Digital Filter

- The allpass delay equalizer is usually designed using a computer-aided optimization method
- We outline one such method next
- Let $H(z)$ be the transfer function of the IIR digital filter with a group delay given by $\tau_H(\omega)$

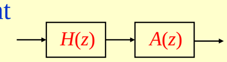
41

Group Delay Equalization of IIR Digital Filter

- Objective is to design a stable allpass section with a transfer function

$$A(z) = \prod_{\ell=1}^{M} \left(\frac{d_{2,\ell} + d_{1,\ell} z^{-1} + z^{-2}}{1 + d_{1,\ell} z^{-1} + d_{2,\ell} z^{-2}} \right)$$

with a group delay $\tau_A(\omega)$ and cascade it with the IIR digital filter $H(z)$ such that the overall group delay $\tau(\omega) = \tau_H(\omega) + \tau_A(\omega)$ is a constant



42

Group Delay Equalization of IIR Digital Filter

- Moreover, to guarantee stability we need to ensure that

$$|d_{2,\ell}| \le 1, \qquad |d_{1,\ell}| < 1 + d_{2,\ell}$$

- The allpass delay equalizer design problem can be formulated as a minimax optimization problem in which we minimize the peak absolute value of the error

$$\mathcal{E}(\omega) = \tau(\omega) - \tau_o$$

Group Delay Equalization of IIR Digital Filter

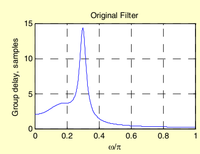
- The adjustable parameters are the desired delay τ_o and the coefficients $d_{1,\ell}, d_{2,\ell}$ of the allpass filter

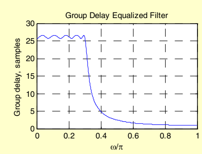
- The M-file `iirgrpdelay` can be used to design the allpass delay equalizer

Group Delay Equalization of IIR Digital Filter

- **Example** – We design an 8-th order allpass equalizer to equalize the group delay of a 4-th order elliptic lowpass filter with a passband edge at 0.3π, passband ripple of 1 dB and a minimum stopband attenuation of 30 dB using Program 9_4.m
- The group delays of the lowpass filter and the overall cascade are shown on the next slide

Group Delay Equalization of IIR Digital Filter





It can be shown that the designed allpass filter is stable