Faculty of Engineering Chulalongkorn University

2102-502 Rand Signal & System, Final Exam, 2 Oct 2007, 9:00-12:00

1) Let $\phi_X(\omega) = \dfrac{(169 - 120\cos\omega)}{(5 - 4\cos\omega)(10 - 6\cos\omega)}$ be a power spectral density of random signal X(t).

   a) Find the innovation filter transfer function H(Z) of X(t). (4)
   b) Find the impulse response h(t) of this filter. (4)
   c) What is the average power of X(t)? (2)

2) If $C_{XX}(\tau) = 4 - |\tau|$, for $\tau = 0, \pm1, \pm2$ and zero elsewhere. Show that x(t) can not exist. (8)

3) Let X(t) be a stationary zero mean continuous-time Gaussian process with power spectral density $\phi_{XX}(\omega) = \dfrac{\omega^2}{1+\omega^4}$.

   a) Find the probability density function of random variable X(5). (5)
   b) Find the transfer function of the whitenning filter of X(t). (5)

4) Proof that the Kahunen-Loeve expansion coefficients are uncorrelated. (8)

5) Brownian Process B(t) will have a covariance function $C_{BB}(s, t) = \min(s, t)$ for $0 \le s, t \le 1$ with eigenvalue $\lambda_n = \left(\dfrac{2}{(2n-1)\pi}\right)^2$, n = 1, 2, 3, … .

   a) Find the total signal energy of B(t) for t = 0 to 1. (4)
   b) If we use only the first 3 terms of the Karhunen-Loeve expansion for approximating B(t) for t = 0 to 1, find the energy of the estimation error. (2)
   c) If we want to have the estimation error energy less than 5% of the total signal energy, how many terms of K-L expansion must be used. (2)

6) Let X(t) be a zero mean random signal defined over the interval $-1 \le t, s \le 1$ with auto-covariance function $c_{XX}(t,s) = 1 + 6ts + 5(3t^2-1)(3s^2-1)$. For Kahunen-Loeve expansion,

   a) use Mercer's theorem to find all *orthonormal* eigenfunctions $\phi_n(t)$ and corresponding eigenvalues λ_n. What is the average signal energy of X(t)? (7)
   b) If one realization is X(t) = t, for $-1 \le t \le 1$, find all its K-L coefficients α_n. (3)

7) Let z(t) and $z'(t)$ be the distance and velocity of a particle, start moving from origin with velocity 10 *m/s*. We can measure $y(t) = z'(t) + N(t)$, the velocity at t = 1, 2, 3, … *s* with zero mean unit white Gaussian measurement noise N(t). But these measurements will change the velocity randomly with zero mean u.w.g.n. V(t), *i.e.*, $z'(t+1) = z'(t) + V(t)$, for t = 1, 2, 3, … *s*. If the velocity changes gradually between measurements, the particle will travel by $\frac{1}{2}\left(z'(t+1) + z'(t)\right)$ *m* during each second.

   a) Using the state vector $\underline{\mathbf{X}}(t) = [\ z(t)\ \ z'(t)\ ]^T$, formulate this problem as a Kalman filter and evaluate (*2x2 matrix*) **A**(t), $\mathbf{P_0}$, (*2x1 vector*) $\underline{\mathbf{B}}$(t), $\underline{\mathbf{h}}$(t) and (*scalar*) Q(t), R(t). (6)
   b) Find (*2x2 matrix*) **M**(t), **P**(t) and (*2x1 vector*) $\underline{\mathbf{k}}$(t) (Kalman gain) at t = 1 and 2 *s*. (6)
   c) If y(1) and y(2) are 11 and 12 *m/s* respectively, find the estimation $\hat{z}(2)$ and $\hat{z}'(2)$. (4)